New concepts in particle physics from the solution of an old problem

2000 J. Phys. A: Math. Gen. 33 5231

(http://iopscience.iop.org/0305-4470/33/29/307)

J. Phys. A: Math. Gen. **33** (2000) 5231–5266. Printed in the UK PII: S0305-4470(00)09882-6

New concepts in particle physics from the solution of an old problem

Bert Schroer†
Institut für Theoretische Physik, FU-Berlin, Arnimallee 14, 14195 Berlin, Germany
E-mail: schroer@cbpf.br

Received 25 November 1999, in final form 13 April 2000

Abstract. Recent ideas on modular localization in local quantum physics are used to clarify the relation between on- and off-shell quantities in particle physics; in particular, the relation between on-shell crossing symmetry and off-shell Einstein causality. Among the collateral results of this new non-perturbative approach are profound relations between crossing symmetry of particle physics and Hawking–Unruh-like thermal aspects (KMS property, entropy attached to horizons) of quantum matter behind causal horizons, aspects which hitherto were exclusively related to Killing horizons in curved spacetime rather than with localization aspects in Minkowski space particle physics. The scope of this modular framework is amazingly wide and ranges from providing a conceptual basis for the $d = 1 + 1$ bootstrap-formfactor programme for factorizable $d = 1 + 1$ models to a decomposition theory of quantum field theories in terms of a finite collection of unitarily equivalent chiral conformal theories placed a specified relative position within a common Hilbert space (in $d = 1+1$ a holographic relation and in higher dimensions more like a scanning). The new framework gives a spacetime interpretation to the Zamolodchikov algebra and explains its thermal aspects.

1. Introduction

Theoretical physicists, in contrast to mathematicians, rarely return to their old unsolved problems; often they replace them by new inventions. The content of this paper on some new concepts in particle physics does not follow this pattern. The old problems it addresses and partially solves are those of the relation between off- and on-shell quantities (or between fields and particles) and, in particular, of crossing symmetry in local quantum physics (LQP)‡. A more restricted form of on-shell crossing symmetry also led to the invention of the dual model and string theory, a line of development which we will not follow except for some remarks in the last section.

The most prominent of the on-shell quantities is the S-matrix of a local quantum field theory (QFT), whereas fields and more general operators are 'off-shell'. The derivation of on-shell quantities from LQP through the use of the rigorous LSZ scattering theory was one of the high points of QFT of the 1960s. In the opposite direction the problem ('*the* inverse problem of

† Present address: CBPF, Rua Dr Xavier Sigaud, 22290-180 Rio de Janeiro, Brazil.
‡ We will often use the name 'local quantum physics' instead of QFT [1], if we have in mind the physical principles of QFT implemented by different concepts from those of the various quantization formalisms (canonical, quantization via path integrals, etc) which most of the readers are familiar with from the various textbooks. To the extent that the reader does not automatically identify QFT with those formalisms, he may continue to use the name QFT without danger of misunderstanding.

0305-4470/00/295231+36$30.00

QFT') lay dormant for a long time. Recently, the adaptation of the Tomita–Takesaki modular theory to wedge-localized algebras has suggested new methods to construct unique off-shell local operator algebras from the scattering data in a quite interesting and novel way [21, 50]. Thus the inverse problem of QFT has a much better status than in quantum mechanics (QM). To bring this into the open requires the introduction of a wealth of new concepts relevant to particle physics, while maintaining all the principles of QFT.

In this paper we will have to study a new kind of operators which, as a result of their weak semi-infinite (wedge-like) localization and their close relation to the S-matrix, are to be considered as on-shell. These on-shell operators are essential for our new approach which avoids pointlike fields at the beginning and rather starts with on-shell generators of wedge-localized algebra which encode the structure of the S-matrix. Off-shell *compactly* localized operators and local field generators are then obtained via intersections of wedge algebras. Here and in the following the word localization region always stands for the causal completion of a spacetime region; these are typically the regions which one obtains by intersecting wedges.

Besides these two extremes there are intermediate possibilities where on-shell and off-shell aspects appear together. The most prominent and useful mixed objects are bilinear forms on scattering vector states, i.e. matrix elements of local operators A (either pointlike fields or bounded operators localized in smaller than wedge regions) taken between incoming and outgoing multiparticle scattering states (in terms of Feynman graphs, one leg is off-shell):

$$^{\text{out}}\langle q_1, \dots q_{n-1}, q_n \,|A|\, p_n, p_{n-1}, \dots p_1\rangle^{\text{in}} \tag{1}$$

which we will call (generalized) formfactors, following the standard terminology of $d = 1+1$ factorizing models. These objects fulfil the important crossing symmetry which acts on the on-shell momenta.

$$\begin{aligned} &^{\text{out}}\langle q_1, \dots q_{n-1}, q_n \,|A|\, p_n, p_{n-1}, \dots p_1\rangle^{\text{in}} \\ &\qquad = {}^{\text{out}}\left\langle -\bar{p}_1, q_1, \dots q_{n-1}, q_n \,|A|\, p_n, p_{n-1}, \dots p_2\right\rangle^{\text{in}} \end{aligned} \tag{2}$$

where the analytic continuation $p \to -p$ is carried out in the rapidity parametrization by an iπ-shift: $\theta \to \theta + \mathrm{i}\pi$, and the bar denotes the antiparticle. The difficulties in physical interpretation of this relation (about which sufficiently general and rigorous information outside of perturbation theory is scarce) reflects the lack in conceptual understanding. It is in a way deeper than the TCP symmetry, a symmetry derived from causality which among other things requires the existence of an antiparticle for each particle. In fact, the crossing transformation is a kind of individual TCP transformation which affects only one particle at a time within the multiparticle incoming ket configuration and carries it to the outgoing bra configuration as an antiparticle. In spite of its name, it is not a quantum theoretical (Wigner) symmetry, since that crossing process involves an on-shell analytic continuation $p \to -p$. Together with vacuum polarization, it belongs to the most characteristic aspects of QFT. Although its physical meaning in terms of the basic principles of LQP remained vague, most physicists liked to view it as a kind of on-shell imprint of Einstein causality, the latter being an off-shell concept. One of the results of the new conceptual framework presented here is an interpretation of the crossing property in terms of 'wedge localization' and the ensuing thermal Hawking–Unruh properties. They are usually associated exclusively with black hole quantum physics, but, in fact, turn out to be general properties of any local quantum description including particle physics in Minkowski space. Although in constructive terms the control in passing from the on-shell S-matrix-dominated aspects to the off-shell local quantum physics remains a formidable problem, it is easy to see that a local theory is (if it exists at all) uniquely determined in terms of its on-shell 'shadow'.

The S-matrix whose matrix elements result from the previous formula for $A = 1$, is *the* observable of particle physics par excellence; it is totally intrinsic and independent of any field coordinatizations, although in the LSZ theory it is calculated from specific fields. Strictly speaking in high-energy physics only (inclusive) cross sections and not amplitudes are directly measured; a fact which is especially important if interactions between zero-mass particles leads to infrared problems.

The reason why most theoretical methods in particle physics do not aim directly at the S-matrix is that most of our physical intuition about causality and charge flows in spacetime is based on (off-shell) local fields or local observables. The new on-shell wedge-algebra generators introduced in this paper are somewhat hidden and, in particular, are not obtainable by Lagrangian or more generally by any kind of quantization approach†. Although their role in general QFT is only at the initial stages of being understood, there is already a very good spacetime comprehension in the class of factorizing $d = 1 + 1$ models [21].

The old problems on which there has been significant recent progress can be summarized in terms of the following questions.

- Does a physically admissible S-matrix fulfilling unitarity, crossing symmetry and certain analytic properties (needed in its formulation), have an underlying unique local QFT? This one may call the *inverse problem of QFT associated with scattering.* It is a problem of particular interest to take note not only of the well known fact that fields and local observable lead to scattering, but also that local equivalence classes of fields‡ or nets of local observables are, in turn, determined by particle scattering data.
- Is there a constructive procedure in which, similar to the $d = 1 + 1$ bootstrap-formfactor programme for factorizing $d = 1 + 1$ models (which, in fact, reappears as a special case), the S-matrix and the generalized formfactors enter as important constructive elements in order to obtain off-shell objects such as fields or local observables? In particular, can one formulate such a constructive approach in a conceptually intrinsic manner, i.e. without any quantization parallelism to classical field theory and without the use of field coordinatizations and short-distance divergence problems? This could be of tremendous practical importance.

The progress obtained on both questions by the modular method will be presented in the following.

In order to better understand what is meant by the word 'old' in the title of this paper, it is very instructive to pause and take stock of some past achievements and failures in an S-matrix approach to particle physics. Already as long ago as 1946 Heisenberg [8] proposed to do particle physics in a pure S-matrix setting in order to avoid the at that time nonsensical aspects of short-distance divergences in QFT. His requirements of unitarity, Poincaré invariance and some rudimentary aspects of cluster decomposition properties turned out to be much too general in order to be useful. A second attempt with the full backing of renormalized perturbation theory was launched in the early 1960s [9]. Part of the motivation was similar to the previous one. Although renormalization theory meanwhile allowed one to extract a class of perturbatively finite QFTs, the formally infinite intermediate steps and the not entirely natural but rather

† Any approach which leads to an explicit solution and not just to formal representations as, for example, Euclidean functional integral representations would of course present all properties. However, the Lagrangian approach only achieves the latter.

‡ That an S-matrix cannot determine individual fields had been known since the late 1950s.

technical looking division into renormalizable/non-renormalizable models nourished the hope that those discomforting features would disappear in a pure S-matrix approach [11]. The second more pragmatic motivation was the idea that the dispersion theoretical research of the 1950s could be extended into a computational scheme for strong interactions within an S-matrix setting. Since the physical principles of the S-matrix approach cannot be dealt with directly but rather require an operator or functional formalism, several ideas which could not be motivated through QFT (in fact, they are not true in perturbative QFT) had to be added. The most prominent ones were on-shell spectral representations as the Mandelstam representation and a strengthened form of crossing called duality. The first one served to specify analyticity domains and the second implemented the purely phenomenological idea of saturating the coexistence of charges in the different crossed channels already by disregarding cuts and only taking one-particle poles into account ('nuclear democracy', 'Reggeization'). I believe that these added requirements which made heavy use of analyticity contributed to the failure of the programme. It did not even achieve the reproduction of those perturbative S-matrices which via LSZ scattering theory were obtained from the Feynman perturbation theory of time-ordered functions. What was left over from this programme got swept aside by the ascending gauge theory at the beginning of the 1970s.

For the purpose of a good understanding of the content of this paper it is very helpful to localize the cause of the failure of the S-matrix bootstrap programme. I think I will be in agreement with most of my colleagues who followed these developments or later read about them that free-floating (and often ill-defined) analyticity requirements are too fine instruments in order to harmonize with physical intuition. Only analytic properties which appear directly in the formulation of physical concepts are useful for the construction of theories. This is best illustrated by two examples. The x-space analyticity of correlation functions in QFT which was discovered by Wightman [5] is equivalent to the spectral, covariance and locality properties of the operator theory. On the other hand, the dispersion relations, even if restricted to the simplest case of forward scattering, involve analyticity properties which arise from a quite complicated interplay between the off-shell causality of retarded functions with on-shell spectrum properties [6]. Such non-constructive analytic properties are still useful for experimentally verifying particular consequences of causality and they also have their merits in the study of possible non-perturbative high-energy bounds on cross sections, but they have no natural role in an actual construction.

Only very few people took note of the fact that the bootstrap programme finally worked in the more limited context of $d = 1 + 1$ integrable models; it was too far away from a 'theory of everything' which was on the minds of the ambitious protagonists of the four-dimensional bootstrap which finally ended in failure. The modest two-dimensional programme led to a nice classification of families of factorizing elastic S-matrices (thus showing that the idea of the bootstrap being a theory of everything (TOE) was incorrect) and it also set the path for the construction of associated QFT models via a formfactor programme. A side result of the S-matrix research in $d = 1 + 1$ was the discovery of an on-shell perturbation theory which, if specialized to on-shell tree graphs without particle creation† allowed one to show the absence of creation of the on-shell one-loop approximation [10]. Apparently, the extension to multi-loops was never elaborated in sufficient generality. The very existence of these formulae shows that a finite on-shell approach which avoids the characteristic off-shell short-distance problems of QFT is more than just a nice dream.

† The absence of particle creation is not an issue which is evident on the level of tree graphs since it only happens on-shell. Properties which are only valid on-shell are too subtle to be seen by inspection of Feynman diagrams.

The present line of research takes off directly where the original programme failed. It removes the unfortunate TOE ideology† from the S-matrix bootstrap and incorporates the latter with the help of modular theory into the QFT structure of wedge algebras. This return into QFT is based on the fact that the S-matrix has in addition to the large time scattering interpretation (well known from the LSZ theory) another little known aspect, namely it is the relative modular invariant between the wedge algebra of incoming free fields and that of the actual interacting Heisenberg operators. Whereas the scattering aspect also applies to QM, the modular role is totally characteristic for local quantum physics. Having established a direct modular relation between the S-matrix and the wedge algebra, the old S-matrix formalism becomes enriched with new physical concepts and mathematical tools. In particular, the thermal aspects of the wedge algebra (the Hawking–Unruh temperature of matter behind a Rindler horizon) becomes inexorably tied up with the crossing symmetry of particle physics. A pivotal role in the linkage of the S-matrix with the wedge algebras is played by special wedge-localized operators which applied to the vacuum create one-particle states without the usually associated cloud of particle–antiparticles well known from vacuum polarization phenomenon. These polarization-free generators (PFGs) cannot exist for spacetime localization regions whose causal completion is smaller than a wedge, unless the theory has no interactions, in other words, the wedge region is the smallest causally complete region for which PFGs are compatible with the presence of interactions.

The new framework shares with the old S-matrix bootstrap programme (and with string theory) the absence of any ultraviolet problems since it uses no coordinatizations in terms of pointlike fields. Whether a theory exists or not is not decided by the short-distance singularities of some field coordinates in terms of which the Lagrangian quantization happened to be done, but rather depends on the non-triviality of the intersection structure of wedge algebras. If intersections representing double-cone algebras contain more operators than just multiples of the identity, the theory is non-trivial in the sense that it possesses a non-trivial net for small localization regions and not just for wedge regions. This avoidance of particular pointlike fields and their short-distance problems was the main dream and the *raison d'être* of the S-matrix bootstrap. It is fully realized in the new approach by the use of the field-coordinate-independent algebraic formulation of QFT (AQFT). The intention of the S-matrix protagonists to abandon fields was reasonable, but unfortunately they thought that they also should abandon the principle of locality.

The philosophy that the S-matrix has nothing to do with localization and locality was also not quite right, as the relation to wedge-localized algebras shows. Since from the net of wedge algebras one can obtain the algebras of compact regions by intersections, all of local quantum physics is in principle determined by the S-matrix. An AQFT for a given S-matrix turns out to be uniquely determined and it is believed that if S is admissible in the old sense (unitary, crossing symmetric + associated analyticity), the net of wedge algebras also exists and leads to the required non-trivial intersections. For $d = 1 + 1$ factorizing S-matrices the formfactor programme goes a long way towards proving this conjecture.

After having outlined our physical motivation and the position of the new concepts with respect to older ideas, we now briefly mention our main mathematical tool which will be used for problems of (quantum) localization: (Tomita's) modular theory of von Neumann algebras‡.

† 'Theories of everything' seem to also be the favourite pastime of post-S-matrix physicists. The underlying idea that certain principles allow for only one solution usually originates in connection with nonlinear structures for which initially no solution is known and ends with too many solutions, thus contradicting the idea of a TOE.

‡ Special aspects (the thermal KMS characterization) of Tomita's mathematical modular theory were discovered by physicists in connection with the quantum statistics of Fermi/Bose systems formulated directly in the infinite-volume limit when the Gibbs formulation breaks down [1].

These concepts, which for the first time clarified the on/off-shell relation and, in particular, the spacetime interpretation of on-shell crossing symmetry, were not available at the time of the S-matrix bootstrap of the 1960. In a seminal paper [12] the connection of wedge-localized algebras with modular theory was established for the first time. The present approach may be considered as the inverse of the Bisognano–Wichmann theorem. Instead of extracting a deep mathematical property from the AQFT of wedge algebras, we are using this property together with the modular role of the S-matrix for the construction of QFTs via wedge algebras. The main new mathematical tool is briefly described in an appendix, a more detailed account can be found in [2].

The ideas of AQFT used in this paper are not as well known as their importance would suggest. Perhaps this is due to the fact that most particle physicists consider QFT as a basically settled issue with only some nasty technical problems remaining. We will demonstrate in this paper that such a view is quite premature and unrealistic.

We have organized this paper as follows. The next section reviews and illustrates the field-coordinate-free approach for theories without interactions and for an interacting $d = 1 + 1$ factorizing model. In the latter case the Zamolodchikov–Faddeev (ZF) algebra emerges in a natural way without having been put in [21], and the hitherto formal ZF operators acquire for the first time a spacetime interpretation in connection with the new PFG generators of wedge algebras. The presentation of these polarization-free wedge generators is extended to systems which are not factorizing (i.e. to theories with on-shell particle creation) in section 3.

After a brief introduction to the AQFT framework in section 4, section 5 treats the light ray/front restriction and algebraic holography in terms of associated chiral conformal field theories. This connection is again a deep result of modular theory, and more specifically of modular inclusions and intersections. There we also discuss the problem of inverting such maps (the 'blow-up' property) in such a way that the original theory becomes reconstructed from a finite number of copies of one abstract chiral theory whose relative position in one Hilbert space has to be carefully chosen. In colloquial terms this is like scanning a higher-dimensional massive theory by one chiral theory in different positions and we will refer to it as 'chiral scanning'. Since a finite number of relatively positioned chiral theories seems to be easier to understand than one higher-dimensional massive theory, the chiral scanning is, in addition, to the wedge algebra method and the use of PFGs, explained before a second potential constructive idea based on modular theory. The mathematical technique used in section 5 is one of the most powerful which AQFT presently is able to offer, namely the theory of modular inclusions and intersections [2, 13].

In the same section we also take up the problem of associating entropy with localized matter. In view of the fact that modular localization leads to thermal aspects which show up in the appearance of a Hawking–Unruh temperature it is natural to ask for a concept of 'localization entropy' and, in the case when it exists, whether it gives a quantum version of the Bekenstein area law for the area of the causal horizon of the localization region.

The final section tries to compare our approach with that of string theory. This is on the one hand natural since both have similar historical roots, but difficult from a conceptual viewpoint because string theory despite all its mathematical formalism has never developed beyond a collection of recipes for formulating underlying principles. Whereas our approach has strong ties with the older S-matrix bootstrap programme (apart from the incorrect TOE philosophy) which only used properties abstracted from QFT, string theory via the dual model has added many *ad hoc* inventions which did not originate through the intrinsic logic of S-matrix theory and QFT and are not asked for by any known principle of particle physics. This is in my view the reason why string theory despite all its semantic changes has remained a collection of computational prescriptions without the guidance of a conceptual framework. Since basic physical issues such as locality and localizations, algebras versus states, etc should be discussed on the level of physical principles and not by looking at formalism and computational recipes, our comparisons in the last section unavoidably remains somewhat vague and superficial.

This presentation is in part a survey of published material [7, 21, 23, 30, 43, 50] as well as of new results, in particular, the presentation of localization entropy in section 4. For the convenience of the reader we attached a two-part appendix, the first part containing standard facts about modular theory and the second one proving that in interacting theories there are no polarization-free generators for subwedge spacetime regions.

2. Systems without interactions and factorizing models

In trying to bring readers with a good knowledge of standard QFT in contact with some new (and old) concepts in algebraic QFT (AQFT) without sending them back with a lot of homework, I face a tricky problem. Let us for the time being put aside the intrinsic logic which would ask for a systematic presentation of the general framework, and let us instead try to manoeuvre in a more less *ad hoc* (occasionally even muddled) way.

In a pedestrian approach the problem of constructing nets of interaction-free systems from Wigner's one-particle theory may serve as a nice pedagogical exercise for the new ideas. The reader who is not familiar with Wigner's representation theoretical method to describe particle spaces is referred to [1, 20]. Since Wigner's representation theory restricted to positive energy representations was the first totally intrinsic relativistic quantum theory without any quantization parallelism to classical particle theory, it is reasonable to expect in general that if we find the right concepts, we should be able to avoid covariant pointlike fields altogether in favour of a more intrinsic way of implementing the causality/locality principle. In that case the local fields should be similar to coordinatizations of local observables in analogy with the use of coordinates in modern differential geometry. This viewpoint is indeed consistent and essential in the present context [21, 22].

Let us first understand how the free-field algebras are directly abstracted from Wigner's theory. By using a spatial variant of Tomita's theory for the wedge situation, i.e. by defining a kind of antilinear involutive 'pre-Tomita' operator $\mathfrak{s}$ on the Wigner representation space (without a von Neumann algebra), one obtains a real closed subspace $H_R(W)$ of the Wigner space H of complex multi-component momentum space wavefunctions as a +1 eigenspace of the Tomita-like quantum mechanical operator $\mathfrak{s}$ in H. Here W denotes the x–t wedge $x > |t|$ and $\mathfrak{s}$ is defined to be the product of the iπ- continued x–t Lorentz boost $\delta^{1/2}$(obtained by the functional calculus associated with the spectral theory of the boost operator $\delta^{\mathrm{i}\tau} := U(\Lambda_{x,t}(\pi\tau)), \pi\tau =$ rapidity) multiplied by the one-particle version of the (antiunitary) j-reflection† in the x–t plane

$$\begin{aligned} \mathfrak{s} &= j\delta^{1/2} \\ s\psi &= \psi \qquad \psi \in H_R(W). \end{aligned} \tag{3}$$

For the definition of the antiunitary Tomita involution j which represents the x–t reflection in the case of antiparticles $\neq$ particles) one needs to extend the Wigner representation to the direct sum of particle/antiparticle spaces; a process well known in the Wigner theory if one wants to include the disconnected Poincaré transformations. Since the x–t reflection commutes with the x–t boost δ^{it} and is antiunitary, it formally inverts the unbounded δ, i.e. $j\delta = \delta^{-1}j$, which is formally the analytically continuing boost at the imaginary value $t = -\mathrm{i}$. As a

† Apart from a rotation around the x-axis by an angle π, this is the famous TCP operator restricted to the one-particle/antiparticle subspace.

result of this commutation relation the unbounded antilinear operator $\mathfrak{s}$ is involutive on its domain of definition $\mathfrak{s}^2 \subset 1$. These unusual properties, which are not met anywhere else in QM, encode geometric localization properties within abstract operator domains [21, 22]. They also pre-empt the relativistic locality properties of QFT which Wigner looked for in vain [14]. The opposite localization, i.e. $H_R(W^{\mathrm{opp}})$, turns out to correspond to the symplectic (or real orthogonal) complement of $H_R(W)$ in H, i.e. $\mathrm{Im}(\psi, H_R(W)) = 0 \curvearrowright \psi \in H_R(W^{\mathrm{opp}})$. Furthermore, one finds the following properties for the subspaces called 'standardness':

$$\begin{aligned} &H_R(W) + \mathrm{i}H_R(W) \text{ is dense in } H \\ &H_R(W) \cap \mathrm{i}H_R(W) = \{0\}. \end{aligned} \tag{4}$$

Having arrived at the wedge localization spaces, one may construct localization spaces for smaller spacetime regions by forming intersections over all wedges which contain this region

$$H_R(\mathcal{O}) = \bigcap_{\mathrm{W} \supset \mathcal{O}} H_R(W). \tag{5}$$

These spaces are again standard and have their own premodular objects δ, j and $\mathfrak{s}$, but this time their action cannot be described in terms of spacetime diffeomorphism. Note also that the modular formalism characterizes the localization of subspaces, but is not able to distinguish individual elements in that subspace. There is a good physical reason for that, because as soon as one tries to do that, one is forced to leave the unique Wigner (m, s) representation framework and pick a particular covariant representation by selecting one specific intertwiner among the *infinite set* of u and v intertwiners which link the unique Wigner (m, s) representation to the countably infinite many covariant possibilities [21]. In this way one would then pass to the framework of covariant fields explained and presented in the first volume of Weinberg's book [20]. The description of a concrete element in $H_R(W)$ or $H_R(\mathcal{O})$ then depends on the choice of covariant formalism. A selection by, for example, invoking Euler equations and the existence of a Lagrangian formalism may be convenient for doing particular perturbative computations or as a mnemotechnical device for classifying polynomial interaction densities†, but is not demanded as an intrinsic attribute of physics.

The way to avoid non-unique covariant fields is to pass from Wigner subspaces directly to von Neumann subalgebras of the algebra of all operators in Fock space $B(\mathcal{H}_{Fock})$, i.e. the transition from real subspaces to von Neumann subalgebras in Fock space is well known. With the help of the Weyl (or CAR in the case of fermions) functor $Weyl(\cdot)$ one defines the local von Neumann algebras [21, 22] generated from the Weyl operators

$$\mathcal{A}(W) := alg\,\{Weyl(f) | f \in H_R(W)\} \tag{6}$$

a process which is sometimes misleadingly called 'second quantization'. These Weyl generators have the formal appearance

$$\begin{aligned} &Weyl(f) = \mathrm{e}^{\mathrm{i}a(f)} \\ &a(f) = \sum_{s_3=-s}^{s} \int (a^*(p, s_3) f_{s_3}(p) + \mathrm{h.c.}) \frac{\mathrm{d}^3 p}{2\omega} \end{aligned} \tag{7}$$

i.e. unlike the covariant fields they are independent of the non-unique intertwiners and depend solely on the unique Wigner data. An analogue statement holds for the half-integer spin case for which the CAR functor maps the Wigner wavefunction into the fermionic generators of

† The causal approach permits the transformation of a polynomial interaction from one coordinatization to any other, whereas a formalism using classical actions involving free-field Lagrangians $\mathcal{L}_0$ is restricted to the use of Euler–Lagrange field coordinatizations.

von Neumann subalgebras. The statistics is already pre-empted by the premodular theory on Wigner space [21]. The local net $\mathcal{A}(\mathcal{O})$ may be obtained in two ways, either one first constructs the spaces $H_R(\mathcal{O})$ via (5) and then applies the Weyl functor, or one first constructs the net of wedge algebras (6) and then intersects the algebras.

If we had taken the conventional route via intertwiners and local fields as in [20], then we would have been forced to use Borchers' construction of equivalence classes† in order to see that the different free fields associated with the (m, s) representation with the same momentum space creation and annihilation operators in Fock space are just different generators of the same coherent families of local algebras, i.e. yield the same net. This would be analogous to working with particular coordinates in differential geometry and then proving at the end that the objects of interest are invariant and therefore independent of coordinates.

The implementation of interactions within the framework of nets requires a radical rethinking of the formalism, even if we are only interested in perturbative aspects. The use of the above method for the Wigner one-particle representation and the subsequent introduction of interactions will inevitably force us to reintroduce field coordinates in order to define what we mean by perturbative interactions. In order to avoid the standard approach we therefore have to find a way to introduce interactions directly into wedge-localized multiparticle spaces or wedge algebras.

In order to find out how we can avoid the use of pointlike fields in interacting situations, let us first ask this question in a more limited context. It is well known that there exists a special class of theories in $d = 1 + 1$ in which the S-matrix commutes with the incoming particle number

$$[S_{sc}, \mathbb{N}_{\text{in}}] = 0 \tag{8}$$

and factorizes on multi-particle in-states [16–18]. For this reason these theories are often referred to as factorizing or integrable (since this leads to an infinite number of conservation laws) models. For those one finds that not only can the old bootstrap programme be carried through, but the application of the so-called formfactor programme allows us to compute even the fields in the sense of bilinear forms between in and out states [25, 28]. Let us ignore those bootstrap–formfactor recipes and try find a modular access to these models by implementing the idea of a relativistic particle pair interaction with a very naive ansatz (assuming for simplicity a situation of self-conjugate particles) which formally generalizes the standard creation/annihilation operators. Using rapidities instead of momenta we require

$$\begin{aligned} Z(\theta)Z(\theta') &= S(\theta - \theta')Z(\theta')Z(\theta) \\ Z(\theta)Z^*(\theta') &= S^{-1}(\theta - \theta')Z^*(\theta')Z(\theta) + \delta(\theta - \theta') \end{aligned} \tag{9}$$

with the star-structure determining the remaining commutation relations and the unitarity of S with $S^{-1}(\theta) = \bar{S}(\theta) = S(-\theta)$, etc. Together with $Z(\theta)\Omega = 0$ we can express all Z correlation functions in terms of Ss and the computation of correlation functions proceeds as for free fields, namely by commuting the annihilation operators Z to the right vacuum, e.g.

$$\begin{aligned} \big(\Omega, Z(\theta_4)Z(\theta_3)Z^*(\theta_2)Z^*(\theta_1)\Omega\big) &= S(\theta_2 - \theta_3)\delta(\theta_3 - \theta_1)\delta(\theta_4 - \theta_2) \\ &\quad + \delta(\theta_3 - \theta_2)\delta(\theta_4 - \theta_1). \end{aligned} \tag{10}$$

Although we use the pre-emptive notation Z which refers to the Zamolodchikov–Faddeev algebra‡, there are for the time being no requirements on the coefficients which go beyond

† The class of covariant free fields belonging to the same (m, s) is a linear subclass of the full equivalence class which comprises all Wick polynomials. In analogy with coordinates in differential geometry this subclass corresponds to linear coordinate transformations.

‡ The missing delta-function contribution in Zamolodchikov's original proposal [26] was later added by Faddeev.

those following from the structure of a distributive *-algebra, i.e. crossing symmetry is not required but will result from wedge localization.

One easily sees that instead of postulating commutation relations, we could also have started from the following formula which represents $Z^*Z^*\Omega$ state vectors in terms of corresponding free-field terms

$$Z^*(\theta_2)Z^*(\theta_1)\Omega = \frac{1}{\sqrt{2}}(\chi_{21}a^*(\theta_2)a^*(\theta_1)\Omega + \chi_{12}S(\theta_2-\theta_1)a^*(\theta_1)a^*(\theta_2)\Omega). \tag{11}$$

Here the symbol $\chi_{P(1)\ldots P(n)}$ denotes the characteristic function of the region $\theta_{P(1)} > \cdots > \theta_{P(n)}$. It is easy to see that the inner product agrees with (10); one only has to use the identity

$$\left\{\chi_{12}S(\theta_2-\theta_1) + \chi_{21}\bar{S}(\theta_3-\theta_4)\right\}\delta(\theta_3-\theta_1)\delta(\theta_4-\theta_2) = S(\theta_2-\theta_1)\delta(\theta_3-\theta_1)\delta(\theta_4-\theta_2). \tag{12}$$

In fact, if we had started with a more general two-particle interaction ansatz by allowing the structure of the second equation in (9) to be different say $S^{-1} \to T$, the consistency with (11) would immediately force us to return to $T = S^{-1}$.

The formula for the four-point function suggests the possibility to replace the algebraic ansatz by the following formula for multi-Z^* state vectors:

$$Z^*(\theta_n)\ldots Z^*(\theta_1)\,\Omega = \sum_{\text{perm}} \chi_{P(n)\ldots P(1)}\left(\prod_{\text{transp}} S\right)a^*(\theta_{P(n)})\ldots a^*(\theta_{P(1)})\,\Omega \tag{13}$$

where the product of S-factors in the brackets contains one S for each transposition which expresses the two-body nature of the interaction. The associativity of the Zs, i.e. the Yang–Baxter relation for matrix-valued Ss ensures the consistency of the formula. We call $\theta_{P(1)} > \cdots > \theta_{P(n)}$ the natural order of the multi-Z^* state vector. From the state characterization (13) one can derive the algebraic definition (9).

With these algebraic prerequisites out of the way, let us now return to the physics and investigate the spacetime localization properties of the following Hermitian operators:

$$\begin{aligned} F(\hat{f}) &= \int F(x)\hat{f}(x)\,\mathrm{d}^2x \qquad \operatorname{supp}\hat{f} \in W \\ &= \int_C Z(\theta)\bar{f}(\theta) \qquad Z(\theta - \mathrm{i}\pi) := Z^*(\theta) \\ \hat{f}(x) &= \frac{1}{\sqrt{2\pi}}\int (f(\theta)\mathrm{e}^{-\mathrm{i}px} + \text{c.c.})\,\mathrm{d}\theta \qquad \bar{f}(\mathrm{i}\pi - \theta) = f(\theta) \end{aligned} \tag{14}$$

where C is a path consisting of the upper/lower rim of a $\mathrm{i}\pi$-strip with the real θ-axis being the upper boundary. Whereas the on-shell value of the Fourier transform $f(\theta)$ of $\hat{f}$ is analytic in this strip, the last relation is a notation (since operators by themselves are never analytic in spacetime labels!) which, however, inside expectation values becomes coherent with meromorphic properties. If we take instead of $Z^\#$ free creation/annihilation operators, the corresponding formula $\int_C a(\theta)\bar{f}(\theta)$ represents wedge-localized smeared free fields. Formally, we may write in analogy to free fields

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2\pi}}\int (\mathrm{e}^{-\mathrm{i}px}Z(\theta) + \text{h.c.})\,\mathrm{d}\theta \\ p &= m(\cosh\theta, \sinh\theta) \end{aligned} \tag{15}$$

but we should be aware that the argument x is not related to a pointlike localization in the sense of causality since on-shell fields are local iff they are *bona fide* free fields, i.e. iff the Zs reduce to the standard creation/annihilation operators (see the appendix).

In the following we will prove that the operators $F(\hat{f})$ with $Z^{\#}$ fulfilling (9) are localized in the wedge $x > |t|$ if and only if the *-algebra can be extended to a Zamolodchikov–Faddeev algebra, i.e. iff the coefficients S are crossing symmetric, including the crossing-symmetric bootstrap pole structure. Following our previously introduced terminology [21], we will use the name polarization-free generators for localized operators in interacting QFT whose one time application to the vacuum vector results in a one-particle state vector. It is well known [18] that PFGs with smaller than wedge localization regions (e.g. double cones, spacelike cones) can only exist in theories without interactions, i.e. $\psi^{\mathrm{in}}_{f_1\ldots f_n} = \psi^{\mathrm{out}}_{f_1\ldots f_n}$. For the convenience of the reader we present the argument in an appendix. PFGs, however, always exist in regions in interacting theories if the localization region is a wedge or bigger [31]. The argument is based on modular theory and will be recollected in the next section.

We want to show that the above F can indeed be converted into *bona fide* PFGs and for a proof we have to check the KMS property for the F-correlation functions with the modular generator being the infinitesimal boost K. This property is a prerequisite for any wedge-localized algebra in a Wightman QFT [12]. The KMS property is well known from statistical mechanics and is the substitute for the Gibbs formula which for many quantum systems becomes meaningless in the thermodynamic limit. In the present context its thermal aspects has been discussed in [21]. The desired KMS property for the wedge reads

$$\langle F(\hat{f}_n)\ldots F(\hat{f}_1)\rangle = \langle F(\hat{f}_{n-1})\ldots F(\hat{f}_2)F(\hat{f}_n^{2\pi\mathrm{i}})\rangle \tag{16}$$

where the superscript $2\pi\mathrm{i}$ indicates the imaginary rapidity translation from the lower to the upper rim of the KMS strip.

A rather straightforward calculation based on the previously explained rules for the Zs yields the following result.

Theorem 1 (See [21, 50]). *The KMS-thermal aspect of the wedge algebra generated by the PFGs is equivalent to the crossing symmetry of the S-matrix*

$$\mathcal{A}(W) := alg\left\{F(\hat{f}); supp\ \hat{f} \in W\right\} \quad \Leftrightarrow \quad S(\theta) = S(\mathrm{i}\pi - \theta).$$

Furthermore, the possible crossing symmetric poles in the physical strip of S will be converted into intermediate composite particle states in the GNS Hilbert space associated with the state defined by the correlations on the $\mathcal{A}(W)$-algebra. The latter commutes with its geometric opposite $\mathcal{A}(W^{\mathrm{opp}})$ in the case of $\mathcal{A}(W^{\mathrm{opp}}) = \mathcal{A}(W)' = A\,\mathrm{d}J\mathcal{A}(W)$. A sufficient condition for this is the existence of a parity transformation whose action on $\mathcal{A}(W)$ equals the commutant $\mathcal{A}(W)'$.

Since the Fs are unbounded operators with (particle number) N-bounds which are the same as for free fields, the algebra generated by them is to be understood in the sense that they are affiliated with the von Neumann algebra which they generate.

We recall the proof for the four-point function of Fs which may be obtained as the scalar product of two-particle state vectors (c.t. denotes the F-contraction terms)

$$F(\hat{f}_2)F(\hat{f}_1)\Omega = \iint \bar{f}_2(\theta_2 - \mathrm{i}\pi)\bar{f}_1(\theta_1 - \mathrm{i}\pi)Z^*(\theta_1)Z^*(\theta_2)\Omega + \text{c.t.} \tag{17}$$

$$= \iint \bar{f}_2(\theta_2 - \mathrm{i}\pi)\bar{f}_1(\theta_1 - \mathrm{i}\pi)\{\chi_{12}a^*(\theta_1)a^*(\theta_2)\Omega + \chi_{21}S(\theta_2 - \theta_1)a^*(\theta_2)a^*(\theta_1)\Omega\} + c\Omega \tag{18}$$

and the analogous formula for the bra-vector. The formula needs some explanation. The symbol χ with the permutation subscript denotes as before the characteristic function associated with the permuted rapidity order. The order for the free creation operators a^* is governed by particle statistics. For each transposition starting from the natural order (13), one obtains an S factor†. The Yang–Baxter relation ensures that the various ways of doing this are consistent. For the inner product the S-dependent terms are. Finally, the terms proportional to the vacuum are contraction terms corresponding to the δ-function in (9). For the S-dependent terms in the inner product we obtain

$$\iint f_4(\theta_2) f_3(\theta_1)\{\chi_{21} S(\theta_2-\theta_1) + \chi_{12}\bar{S}(\theta_1-\theta_2)\}\bar{f}_2(\theta_2-\mathrm{i}\pi)\bar{f}_1(\theta_1-\mathrm{i}\pi)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$

$$= \iint f_4(\theta_2) f_4(\theta_1) S(\theta_2-\theta_1)\bar{f}_2(\theta_2-\mathrm{i}\pi)\bar{f}_1(\theta_1-\mathrm{i}\pi)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \tag{19}$$

The analogous computation for the KMS crossed term in (24) gives

$$\iint f_{2'}(\theta_1) f_2(\theta_2) S(\theta_1-\theta_2) f_1(\theta_1-\mathrm{i}\pi) f_{1'}(\theta_2-\mathrm{i}\pi+2\pi\mathrm{i})\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \tag{20}$$

This formula only makes sense if the $F(f)$ operators are restricted in such a way that the $2\pi\mathrm{i}$ translation on them is well defined, i.e. for wavefunctions f which are analytic in a strip of size $2\pi\mathrm{i}$. It is well known that the KMS condition does not hold on all operators of the algebra, but rather on a dense set of suitably defined analytic elements [24]. The S-independent terms which we have not written down are identical to terms in the four-point function of free fields They separately satisfy the KMS property. What remains is to show the identity of (19) and (20). This is done by a $\theta_2 \to \theta_2 - \mathrm{i}\pi$ contour shift in (20) without picking up terms from infinity. Using the denseness of the wavefunctions one finally obtains

$$S(\theta_2-\theta_1) = S(\theta_1-\theta_2+\mathrm{i}\pi) \tag{21}$$

which is the famous crossing symmetry or the $z \longleftrightarrow -z$ reflection symmetry around the point $z_0 = \frac{1}{2}\mathrm{i}\pi$. For the non-self-conjugate situation the crossed particles are antiparticles and the S on the right-hand side has to be modified accordingly.

In physical terms we may say that the wedge structure of factorizing models is that of a kind of relativistic quantum mechanics. This continues to be true if the crossing symmetric S-matrix has poles in the physical strip. In that case the above contour shift would violate the KMS property unless one modifies the multi-Z^* state vector formula (13) by the inclusion of bound states. For the case $n = 2$ (11) this means

$$Z^*(\theta_2)Z^*(\theta_1)\Omega = \left(Z^*(\theta_2)Z^*(\theta_1)\Omega\right)^{\mathrm{scat}} + |\theta, b\rangle\,\langle\theta, b|Z^*(\theta-\mathrm{i}\theta_{\mathrm{b}})Z^*(\theta+\mathrm{i}\theta_{\mathrm{b}})|\Omega\rangle. \tag{22}$$

The bracket with the superscript 'scat' denotes the previous contribution (11), whereas the last term denotes the bound state contribution. The validity of the KMS property demands the presence of this term and determines the coefficient; here θ_{b} is the imaginary rapidity related to the bound state mass. For a detailed treatment which includes the bound state problem, we refer to a future paper. We emphasize again that it is the representation of the F-correlations in terms of the S-matrix and the KMS property of these correlation functions, which via the GNS construction converts the poles in the (possibly matrix-valued) function S into the extension of the Fock space of the as by additional free-field operators. In this way the poles in numerical functions are converted into the enlargement of Fock space in such a way that a few Zs can

† The notation has used the statistics in order to bring the product of incoming fields a_{in} into the natural order say $1, \ldots, n$. The ordering of the Zs encodes the θ-ordering and not the particle statistics. It is connected with the different boundary values of state vectors and expectation values in θ-space in approaching the physical boundary from the analytic region. This is analogous to the association of the $n!$ n-point x-space correlation functions with different boundary values of one analytic 'master function' in the Wightman theory.

describe many more particles. One may call Z 'fundamental' and introduce new Z_{b} and F_{b}s; the latter will, however, be operators which are already associated with the original F-algebra. What needs an extension is the wedge algebra of *incoming* fields. It is very important to note that this apparent quantum mechanical picture is converted into LQP *with vacuum polarization as soon as we, for example, go to double-cone localization*; this will be shown in the following. The extension of the above proof beyond four-point functions is left to the reader.

With this theorem relating wedge localization via the thermal KMS property to crossing symmetry, we have achieved the main goal of this section: to show that the Zamolodchikov–Faddeev algebra which consists of (9) together with the crossing symmetry of its structure function has a deep spacetime interpretation and an associated thermal KMS aspect. In fact, the simplest PFGs which fulfil conservation of the real particle number and have only elastic scattering (possible in $d = 1 + 1$*) are precisely the ZF algebra operators! In a moment we will see that these models have the full interacting vacuum structure (virtual particle non-conservation) with respect to operators from smaller localization regions (e.g. double cones), i.e. we are dealing with a genuine interacting field theory (and not some relativistic quantum mechanics).*

The KMS computation can be immediately extended to 'formfactors', i.e. mixed correlation functions containing in addition to Fs one generic operator $A \in \mathcal{A}(W)$ so that the previous calculation results from the specialization $A = 1$. This is so because the connected parts of the mixed correlation function is related to the various (n, m) formfactors (1) obtained by the different ways of distributing $n+m$ particles for in and out states. These formfactors are described by different boundary values of one analytic master function which is, in turn, related to the various forward/backward on-shell values which appear in one mixed A–F correlation function. We may start from the correlation function with one A to the left and say n Fs to the right and write the KMS condition as

$$\langle AF(\hat{f}_n)\ldots F(\hat{f}_2)F(\hat{f}_1)\rangle = \langle F(\hat{f}_1^{2\pi\mathrm{i}})AF(\hat{f}_n)\ldots F(\hat{f}_2)\rangle. \tag{23}$$

The n-fold application of the Fs to the vacuum on the left-hand side creates, besides an n-particle term involving n operators Z^* to the vacuum (or KMS reference state vector) Ω, contributions from a lower number of Z^*s together with Z–Z^* contractions. As with free fields, the n-particle contribution can be isolated by Wick-ordering†

$$\langle A : F(\hat{f}_n)\ldots F(\hat{f}_2)F(\hat{f}_1) :\rangle = \langle F(\hat{f}_1^{2\pi\mathrm{i}})A : F(\hat{f}_n)\ldots F(\hat{f}_2) :\rangle. \tag{24}$$

Rewritten in terms of A formfactors the n-particle scattering contribution (using the denseness of the $f(\theta)$) reads as

$$\begin{aligned}\left\langle\Omega, AZ^*(\theta_n)\ldots Z^*(\theta_2)Z^*(\theta_1-2\pi\mathrm{i})\Omega\right\rangle \\ &= \left\langle\Omega, Z(\theta_1+\mathrm{i}\pi)AZ^*(\theta_n)\ldots Z^*(\theta_2)\Omega\right\rangle \\ &= \left\langle Z^*(\theta_1-\mathrm{i}\pi)\Omega, AZ^*(\theta_n)\ldots Z^*(\theta_2)Z^*(\theta)\Omega\right\rangle.\end{aligned} \tag{25}$$

Here the notation suffers from the usual sloppiness of physicists' notation: the analytic continuation by $2\pi\mathrm{i}$ refers to the correlation function and not to the operators. For the natural order of rapidities $\theta_n > \cdots > \theta_1$ this yields the following crossing relation:

$$\left\langle\Omega, Aa_{\mathrm{in}}^*(\theta_n)\ldots a_{\mathrm{in}}^*(\theta_2)a_{\mathrm{in}}^*(\theta_1-\pi\mathrm{i})\Omega\right\rangle = \left\langle a_{\mathrm{out}}^*(\theta_1)\Omega, Aa_{\mathrm{in}}^*(\theta_n)\ldots a_{\mathrm{in}}^*(\theta_2)\Omega\right\rangle. \tag{26}$$

The out-scattering notation on the bra-vectors only becomes relevant upon iteration of the KMS condition since the bra Zs have the opposite natural order. The above KMS relation (24)

† Note that as a result of the commutation relation (9), the change of order within the Wick-ordered products will produce rapidity-dependent factors

contains additional information about bound states and scattering states with a lower number of particles. The generalization to the case of antiparticles $\neq$ particles is straightforward. More generally, we see that the connected part of the mixed matrix elements

$$\left\langle a_{\text{out}}^*(\theta_k)\ldots a_{\text{out}}^*(\theta_1)\Omega,\, Aa_{\text{in}}^*(\theta_n)\ldots a_{\text{in}}^*(\theta_{k-1})\Omega\right\rangle \tag{27}$$

is related to $\langle\Omega, AZ^*(\theta_n)\ldots Z^*(\theta_2)Z^*(\theta_1)\Omega\rangle$ by analytic continuation which *a posteriori* justifies the use of the name formfactors in connection with the mixed A–F correlation functions.

The upshot of this is that such an A must be of the form

$$A = \sum \frac{1}{n!}\int_C \cdots \int_C a_n(\theta_1,\ldots\theta_n) : Z(\theta_1)\ldots Z(\theta_n) : \tag{28}$$

where the a_n have a simple relation to the various formfactors of A (including bound states) whose different in–out distributions of momenta correspond to the different contributions to the integral from the upper/lower rim of the strip bounded by C, which are related by crossing. The transcription of the a_n coefficient functions into physical formfactors (27) complicates the notation, since in the presence of bound states there are a larger number of Fock space particle creation operators than PFG wedge generators F. It is comforting to know that the wedge generators, despite their lack of vacuum polarization clouds, nevertheless contain the full (bound state) particle content (as the bound state formalism of local fields). The wedge algebra structure for factorizing models is like a relativistic QM, but as soon as one sharpens the localization beyond wedge localization, the field-theoretic vacuum structure will destroy this simple picture and replace it with the appearance of the characteristic virtual particle structure which separates local quantum physics from quantum mechanics.

In order to see by what mechanism the quantum mechanical picture is lost in the next step of localization, let us consider the construction of the double-cone algebras as a relative commutants of a shifted wedge (shifted by a inside the standard wedge)

$$\begin{aligned}\mathcal{A}(C_a) &:= \mathcal{A}(W_a)' \cap \mathcal{A}(W)\\ C_a &= W_a^{\text{opp}} \cap W.\end{aligned} \tag{29}$$

For $A \in \mathcal{A}(C_a) \subset \mathcal{A}(W)$ and $F_a(\hat{f}_i) \in \mathcal{A}(W_a) \subset \mathcal{A}(W)$ the KMS condition for the W-localization reads as before, except that whenever an $F_a(\hat{f}_i)$ is crossed to the left-hand side of A, we may commute it back to the right-hand side since $[\mathcal{A}(C_a), F_a(\hat{f}_i)] = 0$. The resulting relations are, for example,

$$\langle AF_a(\hat{f}_1) : F_a(\hat{f}_n)\ldots F_a(\hat{f}_2) :\rangle = \langle A : F_a(\hat{f}_n)\ldots F_a(\hat{f}_2)F_a(\hat{f}_1^{2\pi \mathrm{i}}) :\rangle. \tag{30}$$

Note that $F_a(\hat{f}_1)$ is outside the Wick-ordering. Since it acts neither on the bra nor the ket vacuum, it contains both frequency parts. The creation part can be combined with the other Fs under one common Wick-ordering, whereas the annihilation part via contraction with one of the Wick-ordered Fs will give an expectation value of one A with $(n-2)$ Fs. Using the representation (14) for the Fs and going to rapidity space we obtain [23] the so-called kinematical pole relation

$$Res_{\theta_{12}=\mathrm{i}\pi}\left\langle AZ^*(\theta_n)\ldots Z^*(\theta_2)Z^*(\theta_1)\right\rangle = 2\mathrm{i}\boldsymbol{C}_{12}\left\langle AZ^*(\theta_n)\ldots Z^*(\theta_3)\right\rangle(1 - S_{1n}\ldots S_{13}). \tag{31}$$

Here the product of two-particle S-matrices results from commuting the $Z(\theta_1)$ to the right so that it stands to the left of $Z^*(\theta_2)$, whereas the charge-conjugation matrix $\boldsymbol{C}$ only appears if we relax our assumption of self-conjugacy.

I believe that the general issue of the shape of polarization clouds in terms of their asymptotic (say incoming) particle content is intimately related to the structure of the as yet unknown modular automorphisms which exist for each spacetime region.

The relation (31) appears for the first time in Smirnov's axiomatic approach [28] as one of his recipes; more recently it was derived as a consequence of the LSZ formalism adapted to the factorizing model situation [29]. In the present approach it has an apparently very different origin: it is together with the ZF algebra structure a consequence of the wedge localization of the generators $F(\hat{f})$ and the sharpened double-cone locality (29) of A. The existence problem for the QFT associated with an admissible S-matrix (unitary, crossing symmetric, correct physical residua at one-particle poles) of a factorizing theory is the non-triviality of the relative commutant algebra, i.e. $\mathcal{A}(C_a) \neq \boldsymbol{C} \cdot 1$. Intuitively, the operators in double-cone algebras are expected to behave similarly to pointlike fields applied to the vacuum. Namely, one expects the full interacting polarization cloud structure. For the case at hand this is, in fact, a consequence of the above kinematical pole formula since this leads to a recursion which for non-trivial two-particle S-matrices is inconsistent with a finite number of terms in (28). Only if the bracket containing the S-products vanishes, is the operator A a composite of a free field.

The determination of a relative commutant or an intersection of wedge algebras even in the context of factorizing models is not an easy matter. We expect that the use of the following 'holographic' structure significantly simplifies this problem. We first perform a *lightlike translation* of the wedge into itself by letting it slide along the upper light ray by an amount given by the lightlike vector a_+. We obtain an inclusion of algebras and an associated relative commutant

$$\begin{aligned} &\mathcal{A}(W_{a_\pm}) \subset \mathcal{A}(W) \\ &\mathcal{A}(W_{a_\pm})' \cap \mathcal{A}(W). \end{aligned} \tag{32}$$

The intuitive picture is that the relative commutant lives on the $a_\pm$ interval of the upper/lower light ray, since this is the only region inside W which is spacelike to the interior of the respective shifted wedges. This relative commutant subalgebra is a light ray part of the above double-cone algebra, and it has an easier mathematical structure. One only has to take a generic operator in the wedge algebra which formally can be written as a power series (28) in the generators and [21, 50] find those operators which commute with the shifted Fs

$$\left[A, U(e_+)F(f)U^*(e_+)\right] = 0. \tag{33}$$

Since the shifted Fs are linear expressions in the Zs, the nth-order polynomial contribution to the commutator comes from only two adjacent terms in A; namely from a_{n+1} and a_{n-1}, which correspond to the annihilation/creation term in F. The size of the shift gives rise to a Paley–Wiener behaviour in the imaginary direction, whereas the relation between a_{n+1} and a_{n-1} is identical to (31), so we do not learn anything new beyond what was already observed with the KMS technique (30). However, as will be explained in section 5, the net obtained from the algebra

$$\mathcal{A}_\pm := \cup_{a_\pm} A(C_{a\pm}) \tag{34}$$

is a chiral conformal net on the respective subspace $H_\pm = \overline{\mathcal{A}_\pm \Omega}$. If our initial algebra were $d = 1 + 1$ conformal, the total space would factorize $H = H_+ \bar{\otimes} H_- = \overline{\left(\mathcal{A}_+ \bar{\otimes} \mathcal{A}_-\right) \Omega}$, and we would recover the well known fact that two-dimensional local theories factorize into the two light ray theories. If the theory is massive, we expect $H = \overline{\mathcal{A}_+ \Omega}$, i.e. the Hilbert space obtained from one horizon already contains all state vectors. This would correspond to the difference in classical propagation of characteristic massless versus massive data in $d = 1 + 1$. There it is known that although for the massless case one needs the characteristic data on the two light rays, the massive case requires only one light ray. In fact, there exists a rigorous proof that this classical behaviour carries over to free quantum fields: with the exception of $m = 0$ massless theories, in all other cases (including lightfront data for higher-dimensional

$m = 0$ situations) the vacuum is cyclic with respect to one lightfront $H = \overline{\mathcal{A}_\pm \Omega}$ [30]. The proof is representation theoretic and holds for all cases except the $d = 1 + 1$ massless case. Hence in the case of interaction-free algebras the holographic lightfront reduction which, has $d - 1$ dimensions, always fulfils the Reeh–Schlieder property for $d > 2$, where for $d = 1 + 1$ only massive theories obey holographic cyclicity. In order to recover the wedge algebra with its net structure from the holographic restriction, one needs the opposite light ray translation with $U(a_-)$, i.e. $\mathcal{A}(W) = \cup_{a_-<0} A \,\mathrm{d}U(a_-)\mathcal{A}_+$. For the non-triviality of the net associated with $\mathcal{A}(W)$ it is sufficient to show that the associated chiral conformal theory is non-trivial. In order to achieve this, one has to convert the bilinear forms (28) in the Z-basis, which fulfil the recursion relation into genuine operators on the one-dimensional light ray. This is beyond the scope of this paper.

Hence the modular approach leads to a dichotomy of *real particle creation* (absent in factorizing models) in the PFGs and in the aspect of wedge localization, versus the full QFT *virtual particle structure* of the vacuum† for the more sharply localized operators. In some sense the wedge is the best compromise between the particle/field point of view. *In this and only in this sense the particle–field dualism (as a generalization of the particle–wave dualism of QM) applies to QFT*. Since it is left invariant by an appropriate L-boost, the algebra contains enough operators in order to resolve at least the vacuum and one-particle states (which cannot be resolved from the remaining states in any algebra with a smaller localization).

In the next section we will argue that these properties are not freaks of factorizing models, whereas in a later section we will reveal the more mathematical aspects of lightcone subalgebras and holography. As we have argued on the basis of the previous pedestrian approach, the holography aspect will be important in the modular construction of QFTs, because it delegates certain properties of a rather complicated theory to those of (in general several) simpler theories.

It is worthwhile highlighting two aspects which already are visible from these considerations. One is the notion of 'quantum localization' in terms of algebraic intersections as compared with the more classical localization in terms of test function smearing of pointlike fields. As mentioned already, the wedge localization of the PFGs cannot be improved by choosing smaller supports of test functions inside the wedge; the only possibility is to intersect algebras. In that case the old generators become useless, for example, in the description of the double-cone algebras; the latter has new generators. Related to this is that the short-distance behaviour loses its dominant role.

If one does not use field coordinatizations, it is not even clear what one means by 'the (good or bad) short-distance behaviour of a theory'. Short-distance behaviour of what object? There is no short-distance problem of PFGs, since they have some 'natural cut-off' (to the extent that the use of such words which are filled with preassigned old meaning are reasonable in the new context). *The intersection of algebras does not give rise to short-distance problems* in the standard sense of this word. An explicit construction of pointlike field coordinates from algebraic nets is presently only available for chiral conformal theories [33]. It produces fields of arbitrary high operator dimension, and as a result of its group-theoretical techniques it also does not suffer from short-distance problems. The absence of short-distance problems in the modular localization approach seems to be of an entirely different nature from statements about the absence of ultraviolet problems in string theory.

The results in this section should be viewed as an extension of the Wigner theory into the realm of interactions for a special class of models.

† The deeper understanding of the virtual vacuum structure (or the particle content of say state vectors obtained by application of a double-cone localized operator to the vacuum) is presumably hidden in the modular groups of double-cone algebras.

3. PFGs in the presence of real particle creation

For models with real particle creation it is not immediately clear how to construct PFGs. In order to obtain some clue we first look at $d = 1 + 1$ theories which do not have any transversal extension to wedges. Furthermore, we assume that there is only one kind of particle which corresponds to the previous assumption concerning the absence of poles in the two-particle S-matrix for factorizing models. Modular theory always ensures the existence of PFGs [31]. In fact, for every modular localized state vector

$$\begin{aligned} S\psi &= \psi \\ S &= J\Delta^{1/2} \end{aligned} \tag{35}$$

there exists a (generally unbounded) operator G associated with the von Neumann $\mathcal{A}$ algebra in standard position such that

$$G\Omega = \psi \tag{36}$$

$$G^*\Omega = S\psi. \tag{37}$$

For the case at hand $\mathcal{A} = \mathcal{A}(W)$, $\Omega = |0\rangle$ = vacuum, there exists a dense subspace of explicitly constructable (see below) wedge-localized state vectors $\mathcal{H}_R^{\text{Fock}} + \mathrm{i}\mathcal{H}_R^{\text{Fock}}$ which possess affiliated Gs. Since $H_R(W) \subset \mathcal{H}_R^{\text{Fock}}$ there exists an algebra-affiliated operator $F(f)$ for each vector f in $H_R(W) + \mathrm{i}H_R(W)$ with

$$\begin{aligned} F(f)\Omega &= |f\rangle = \text{one-particle state} \\ F(f)^*\Omega &= S\,|f\rangle\,. \end{aligned} \tag{38}$$

Although the existence of PFGs outside of factorizing models poses no problems, the presence of particle creation prevents them from having amenable algebraic properties. The interpretation in the form of

$$F(f) = \int F(x)\hat{f}(x)\,\mathrm{d}^2x \tag{39}$$

$$F(x) = \int \left(Z(\theta)\,\mathrm{e}^{-\mathrm{i}px} + \bar{Z}^*(\theta)\,\mathrm{e}^{\mathrm{i}px}\right)\mathrm{d}\theta \tag{40}$$

where $F(x)$ is a tempered operator-valued distribution on a dense translation-invariant domain which holds in the factorizing case is not compatible with particle creation [31] because it leads to relative commutation relations of F with the incoming/outgoing free field

$$\begin{aligned} \left[Z^\#(\theta), a_{\text{in}}^\#(\theta')\right] &= 0,\ \theta < \theta' \\ \left[Z^\#(\theta), a_{\text{out}}^\#(\theta')\right] &= 0,\ \theta > \theta' \end{aligned} \tag{41}$$

and similarly for the antiparticle operators $\bar{Z}^\#$. Therefore, we will first try to see how far we can get with localized states.

Again we specialize to the self-conjugate case $\bar{Z} = Z$ and the absence of bound states. From the previous discussion we take the idea that we should look for a relation between the ordering of rapidities and the action of the scattering operator. We fix the state vector $\psi(\theta_n, \ldots, \theta_1)$ for the natural θ-order to be an incoming n-particle state as we did for the previous particle-conserving situation. The totally mirrored order should then be a vector obtained by applying the full S-matrix to the incoming n-particle vector.

However, what should we do for the remaining permutations? We should end up with a prescription which for factorizing systems agrees with the previous formalism. For two fs there is no problem; the formula looks as before (11), except that the application of the

S-operator to the two-particle in-vector has components to all n-particle multiparticle vectors for $n \geqslant 2$, i.e. the rapidities are labels which are not related to the incoming particle content of the state vector

$$\psi_2(f_2, f_1) \sim \int\int_C (\chi_{21} a(\theta_2)a(\theta_1) + \chi_{21} S a(\theta_1)a(\theta_2))\,|0\rangle\, \bar{f}_2(\theta_2)\bar{f}_1(\theta_1)\,\mathrm{d}\theta_2\,\mathrm{d}\theta_1$$
$$\times \left\langle 0|a(\theta_n)\ldots a(\theta_3)Sa^*(\theta_1)a^*(\theta_2)|0\right\rangle \neq 0 \qquad n \geqslant 4. \tag{42}$$

The check of the localization equation $J\Delta^{1/2}\psi_2 = \psi_2$ with $J = J_0 S$ (again we omitted the subscript 'scat' from the S-matrix and we used the previous notation where the integration path C includes creation as well as annihilation contributions there will also be a contraction term. The inner product between two ψ_2 turns out to be

$$\int \bar{f}_1'(\theta_1')\bar{f}_2'(\theta_2')\left\langle 0|a(\theta_1')a(\theta_2')a^*(\theta_2)a^*(\theta_1)|0\right\rangle f_2(\theta_2)f_1(\theta_1) + \text{c.t.}$$
$$+\int \bar{f}_1'(\theta_1')\bar{f}_2'(\theta_2')\,\langle 0|\,a(\theta_2')a(\theta_1')(S\chi_{21} + S^*\chi_{12})a^*(\theta_2)a^*(\theta_1\,|0\rangle\, f_2(\theta_2)f_1(\theta_1) \tag{43}$$

where c.t. denotes the contraction term contributions coming from an annihilation part in C and we have disregarded problems of overall normalizations and the integration is done over all θ and θ'. Using the unitarity property of S and the boundary property $\bar{f}(\mathrm{i}\pi - \theta) = f(\theta)$, the last term can be written without the ordering χs as

$$\bar{f}_1'(\theta_1')\bar{f}_2'(\theta_2')\,\langle 0|\,a(\theta_2')a(\theta_1')Sa^*(\theta_2)a^*(\theta_1\,|0\rangle\, f_2(\theta_2)f_1(\theta_1) \tag{44}$$

and has the same form as for the previous factorizing case if one replaces (20). In order to establish the KMS property we have to write this inner product as

$$(\psi_2', \psi_2) = (\psi_1', \psi_3) \tag{45}$$

where ψ_1 is a one-particle vector. So we have to figure out how permutations beyond the natural order and its mirror image are represented on tensor product factors of incoming state vectors. Some thinking reveals that subsequent applications of S-matrices on tensor factors of the n-particle tensor product vectors only makes sense for non-overlapping situations. The action of the S-matrix on one tensor factor is associated with the mirror perturbation of that tensor factor $12\ldots k \rightarrow k\ldots 21$ since intuitively speaking one only obtains the full k-particle scattering if the incoming velocities (or rapidities) are such that all particles meet kinematically, which only happens if the order of incoming velocities is the mirrored natural order. Mathematically, we should write each permutation as the non-overlapping product of 'mirror permutations' The smallest mirror permutations are transpositions of adjacent factors. An example for an overlapping product is the product of two such transpositions which have one element in common, e.g. $123 \rightarrow 132 \rightarrow 312$; there is no meaning in terms of a subsequent tensor S-matrix action. However, the composition $123 \rightarrow 213 \rightarrow 312$ has a meaningful S-matrix counterpart; namely $S \cdot S_{12}a^*(\theta_1)a^*(\theta_2)a^*(\theta_3)\Omega$ where S_{12} leaves the third tensor factor unchanged, i.e. is the Fock space vector $(Sa^*(\theta_1)a^*(\theta_2)\Omega)\otimes a^*(\theta_3)\Omega$ on which the subsequent action of S (which corresponds to the mirror permutation of all three objects) is well defined. In general, if one mirror permutation is completely inside a larger one the scattering correspondence which is consistent with the tensor product structure of Fock space. On the other hand, for overlapping products of mirror permutations the association to scattering data becomes meaningless, where overlapping means that part of each mirror permutation is outside of the other. Fortunately, as it is easy to see, there is precisely one representation in terms of non-overlapping mirror permutations. This leads to a unique representation of multi-f labelled state vectors in terms of scattering data. On the other hand, if we were to write each mirror permutation as a product of (necessarily overlapping) transpositions, we lose the uniqueness and we then need the Yang–Baxter structure in order to maintain consistency; in this case we return to the modular setting of factorizing models in the previous section.

Let us elaborate this in a pedestrian fashion by writing explicit formulae for $n = 3$. The state vector is a sum of $3! = 6$ terms

$$\begin{aligned}\psi_3(f_3, f_2, f_1) \simeq \iiint_C &\{\chi_{321} a(\theta_3)a(\theta_2)a(\theta_1) + \chi_{312} S_{21} a^*(\theta_3)a^*(\theta_2)a^*(\theta_1)\\ &+\chi_{231} S_{32} a(\theta_3)a(\theta_2)a(\theta_1) + \chi_{123} S_{321} a(\theta_3)a(\theta_2)a(\theta_1)\\ &+\chi_{132} S_{321} \cdot S_{23}^* a^*(\theta_3)a(\theta_2)a(\theta_1)\\ &+\chi_{213} S_{321} \cdot S_{12}^* a(\theta_3)a(\theta_2)a(\theta_1)\} |0\rangle\, \bar{f}_3(\theta_3)\bar{f}_2(\theta_2)\bar{f}_1(\theta_1).\end{aligned}$$

Here χ denotes again the characteristic function of the respective θ-orders and $S_{\ldots}$ acts on the respective tensor factor with the remaining particle being a spectator. As before one checks that this vector fulfils the modular localization equation $SJ_0\Delta^{1/2}\psi_3 = \psi_3$; the Tomita operator acting on ψ_3 just reshuffles the six terms. As in the two-particle case, this action creates a vector with a complicated incoming particle content having components to all particle numbers. The last two terms correspond to nested mirror permutations and, as will be seen below, results in the appearance of 'non-diagonal inclusive processes' terms in the (ψ_3', ψ_3) inner product which generalize the diagonal inclusive processes [7] which result from the summation over final states in cross sections.

As an example we write down the integrand of one of those non-diagonal inclusive terms

$$\left\langle 0 \left| a(\theta_1')a(\theta_2')a(\theta_3') S \cdot S_{12}^* a^*(\theta_3)a^*(\theta_2)a^*(\theta_1) \right| 0 \right\rangle. \tag{46}$$

In a graphical scattering representation particles 1 and 2 would scatter first and produce arbitrarily many (subject to the conservation laws for the total energy–momentum) particles which together with the third incoming particle (which hitherto was only a spectator) enter an additional scattering process of which only the three-particle outgoing component is separated out by the matrix element in (46). The dot denotes summation over all admissible intermediate states and could be represented by, for example, a heavy line in the graphical representation in order to distinguish it from the one-particle lines. We will not write down the six contributions to the inner product coming from the creation part and the remaining ones involving the annihilation parts from the path C.

Our main interest is the study of (ψ_1', ψ_3),

$$\begin{aligned}(\psi_1', \psi_3) &= \int_C \left\langle 0 \left| a(\theta_1')\{1\chi_{321} + S\chi_{123} + S \cdot S_{12}\chi_{213}\} a(\theta_3)a(\theta_2)a(\theta_1) \right| 0 \right\rangle\\ &\quad\times \bar{f}_1'(\theta_1')\bar{f}_3(\theta_3)\bar{f}_2(\theta_2)\bar{f}_1(\theta_1)\\ &= \int \left\langle 0 \left| a(\theta_1')\{1\chi_{321} + S + S \cdot S_{12}\chi_{213}\} a(\theta_3)a^*(\theta_2)a^*(\theta_1) \right| 0 \right\rangle\\ &\quad\times \bar{f}_1'(\theta_1')\bar{f}_3(\theta_3) f_2(\theta_2) f_1(\theta_1)\\ &= \int \left\langle 0 \left| a(\theta_1')a(\theta_3)\{1 + S\} a^*(\theta_2)a^*(\theta_1) \right| 0 \right\rangle \bar{f}_1'(\theta_1')\bar{f}_3(\theta_3) f_2(\theta_2) f_1(\theta_1).\end{aligned} \tag{47}$$

The last line is (apart from an $a(\theta_2)$–$a^*(\theta_1)$ contraction term) the only non-vanishing contribution. Here the S-factor in front of S_{12} has been transferred as S^* onto the left-hand one-particle vector whereupon it acts as the identity. Renaming $f_3(\theta_3) \to f'(\theta_2')$ we obtain the result (43). We now apply the KMS property for inner products of modular subspaces

$$\begin{aligned} &(\psi', \psi) = (\psi, \Delta\psi') \\ &\psi, \psi' \in \mathcal{H}_R(W) \subset \mathcal{H}_{\text{Fock}}. \end{aligned} \tag{48}$$

For the case at hand $\psi' = \psi_1', \psi = \psi_3$ the particle interpretation of this KMS relation for modular vectors is precisely the crossing symmetry relation. For the more general case antiparticles $\neq$ particles, one has to work in the dense complex subspace $\mathcal{H}_R(W) + \mathrm{i}\mathcal{H}_R(W)$ (which is a complete Hilbert space in its own right in the thermal topology [21]. The conversion of the KMS property for the inner product of the modular localized state vectors with $n-1$ f labels with a one-particle vector containing one f contains the crossing information for scattering of $n_{\text{in}} + n_{\text{out}} = n$ particles.

The crucial question is whether these inner products can also be used in order to define correlation functions of PFG n-point functions

$$\langle 0 | F(f_1) \ldots F(f_n) | 0 \rangle \tag{49}$$

for the lowest non-trivial case of a four-point function we have already checked one such condition (45). I have carried out other consistency checks and do not think that the prerequisites of [31] leading back to particle conservation can be derived from these correlation functions. The question of whether and how these would be correlation functions are related to the perturbative on-shell S-matrix representations mentioned in the introduction is particularly interesting and I hope to return to the issue of the form of PFG correlation functions in a more complete and systematic way in a future paper.

For non-factorizing theories the interest in the modular localization approach is (besides the improvement in the understanding the structure of interacting QFT) the possible existence of an on-shell perturbation theory of local nets, avoiding the use of the non-intrinsic field coordinatizations. This is a revival of the perturbative version of the old dream to construct S-matrices just using crossing symmetry in addition to unitarity and no pointlike fields. The old S-matrix bootstrap programme admittedly did not get far, but now we perhaps can formulate a similar but structurally richer problem as a perturbative approach to correlation functions of the on-shell PFGs. Modular theory has given us a lot of insight and nobody nowadays would try to cleanse the Einstein causality and locality concepts from the stage as was done in the 1960s. On the contrary, the local off-shell observable algebras would be at the centre of interest and the avoidance of quantization would have entirely pragmatic reasons. In particular, the sharpening of localization beyond wedges is done by algebraic intersections of wedge algebras rather than by cut-off or test function manipulations on field coordinates.

The successful $d = 1 + 1$ bootstrap–formfactor programme of the previous section for factorizing models yields S-matrices and formfactors which for models with a continuous coupling are analytic around $g = 0$. A good illustration is the sine–Gordon theory [29]. The more local off-shell quantities, however (i.e. pointlike field operators or operators from algebras belonging to bounded regions), are radically different since they involve virtual particle polarization clouds which formally may be represented by infinite series in the on-shell Fs similar to the factorizing $d = 1+1$ case of the previous section. The analytic status of these quantities (i.e. localized operators and their correlation functions) is presently not known; it may well turn out that they are only Borel summable or (in the general non-factorizable case) worse. The on-shell/off-shell dichotomy of the modular approach for the first time allows us to determine more precisely whether the cause of the possible breakdown of analyticity at $g = 0$ are the polarization clouds.

A solution to these problems, even if limited to some new kind of perturbation theory (a perturbation theory of wedge algebras and their intersections) should also shed some light on the question of how to handle theories involving higher-spin particles, which in the standard off-shell causal perturbation theory lead to short-distance non-renormalizability. A very good illustration of what I mean is the causal perturbation of massive spin $= 1$ vector mesons. Here the coupling of covariant fields obtained by covariantizing the Wigner particle representation theory in the sense of the previous section will not be renormalizable in the sense of short-distance power counting. In the standard perturbative approach the indefinite metric ghosts are used to lower the operator dimension of the interaction densities (free-field polynomials) $W(x)$, which as a result of the free vector meson dimension $\dim A_\mu = 2$, are at least 5, down to the value 4 permitted by the renormalization requirements in a $d = 1 + 3$ causal perturbative approach [38]. Since the ghosts are removed at the end, the situation is akin to a catalyser in chemistry: they do not appear in the original question and are absent in the final result (without leaving any intrinsic trace behind). In theoretical physics the presence of such catalysers should be understood as indicating that the theory needs to be analysed on a deeper level of local quantum physics, i.e. further away from quantization and quasiclassics. Indeed, in the present on-shell modular approach the short-distance operator reason for introducing such ghosts would not be there and the remaining question is again whether the modular programme allows for a perturbative analytically manageable formulation.

4. The AQFT framework

After our pedestrian presentation of the wedge algebra approach, it is time to be more systematic and precise. For a non-interacting free system the conversion of the rather simple spatial nets of real subspaces of the Wigner space of momentum space (m, s) wavefunctions into an interaction-free net in Fock space produces the following three properties which continue to hold in the presence of interactions. They have been explained in many papers [40] and in a textbook [1], and my main task here is to adapt them to the problems of this paper.

(a) *A net of local* (C^*- or von Neumann) *operator algebras* indexed by classical spacetime regions $\mathcal{O}$

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}).$$

Without loss of generality the regions $\mathcal{O}$ may be restricted to the Poincaré covariant family of general double cones and the range of this map may be described in terms of concrete operator algebras in Hilbert space for which the vacuum representation π_0 may be taken, i.e. $\mathcal{A}(\mathcal{O}) \equiv \pi_0(\mathcal{A}(\mathcal{O}))$. The geometrical and physical coherence properties as isotony, $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$, and Einstein causality, $\mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O})'$, are then evident coherence requirements. Here we use the standard notation of AQFT: the prime on a region denotes the causal disjoint and on the von Neumann algebra it denotes the commutant within $B(H)$, where H is the ambient Hilbert space (here the representation space of the vacuum representation). Einstein causality can be interpreted as an *a priori* knowledge about some with $\mathcal{A}(\mathcal{O})$ commensurable observables in the sense of von Neumann. This causality property suggests the question whether complete knowledge about commensurability $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$ is possible. It turns out that this is indeed the generic behaviour of vacuum nets called Haag duality. The cases of violation of this duality are of particular interest since they can be related to a very fundamental intrinsic characterization of spontaneous symmetry breaking, thus vastly generalizing the Nambu–Goldstone mechanism which was first found with Lagrangian quantization [1].

(b) *Poincaré covariance and spectral properties.*

$$g \in \mathcal{P} \to \alpha_g \text{ automorphism}$$

$$\alpha_g(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O})$$

is unitarily implemented in the vacuum representation

$$U(g)AU^*(g) = \alpha_g(A)$$

$$A \in \mathcal{A}(\mathcal{O}).$$

The unitaries for the translations have energy–momentum generators which fulfil the relativistic spectrum (positive energy) condition, symbolically $specU(a) \in \overline{V^\uparrow}$ (the closed forward lightcone.)

(c) *The phase-space structure of local quantum physics or the 'nuclearity property'.*

Remark 2. The precise formulation of the third property is somewhat involved and will be presented after the following remarks on the first two structural properties. Since in the formulation of the net one may, without loss of generality, work with von Neumann algebras [1], the first question is what type of Murray–von Neumann–Connes–Haagerup classification occurs. There is a very precise answer for wedges (which may be considered as double cones at infinity). As a result of the existence of a one-sided translation into a wedge as well as of the split property below, the wedge algebras $\mathcal{A}(W)$ turn out to be a hyperfinite factor of type III_1. This implies, in particular, that the algebra has properties which take it far away from the structure of QM (factors of type I_∞). Such algebras do not have pure states or minimal projectors, rather all faithful states on such algebras are thermal, i.e. obey the KMS condition. This makes them similar to states appearing in CST with bifurcated horizons as in Hawking–Unruh situations, however, with modular flows instead of Killing flows (but more 'quantum', i.e. without the classical geometric Killing vector aspects of horizons). The modular flow near the boundary of, for example, double-cone regions become asymptotically geometric and Killing-like.

The nuclearity requirement results from the idea that there should be a local quantum physical counterpart of the phase-space properties of QM in a box. The famous *finite number of degrees of freedom law per unit cell of QM phase space* results from limiting the discrete box spectrum by a cut-off in energy. As first suggested by Haag and Swieca [1], the corresponding LQP counterpart, based on the causally closed double-cone analogue of the quantization box in Schrödinger QM, points in the direction of a 'weakly' infinite number; according to their estimates this set of state vectors was compact in Hilbert space. Subsequent refinements of techniques revealed that this set is slightly smaller, namely 'nuclear' [1], and exact calculations with interaction-free theories demonstrated that the phase-space situation also cannot be better than nuclear.

The best way to understand this issue is to follow the motivating footsteps of Haag and Swieca. They, and many other physicists at that time (and as some contemporary philosophers [37]), were attracted by the intriguing consequences of the so-called Reeh–Schlieder property of QFT

$$\begin{aligned} &\overline{\mathcal{P}(\mathcal{O})\Omega} = H \qquad \text{cyclicity of } \Omega \\ &A \in \mathcal{P}(\mathcal{O}) \qquad A\Omega = 0 \implies A = 0 \qquad \text{i.e.} \quad \Omega \text{ separating} \end{aligned} \tag{50}$$

which either holds for the polynomial algebras of fields (which are affiliated to the von Neumann algebras which they generate) or for operator algebras $\mathcal{A}(\mathcal{O})$. The first property, namely the denseness of states created from the vacuum by operators from arbitrarily small localization regions (e.g. a state describing 'a particle behind the moon'† and a charge-compensating antiparticle in some other far-away region can be approximated inside a laboratory of arbitrary small size and duration) is totally unexpected from the global viewpoint of general QT. In the algebraic formulation this can be shown to be dual to the second one (in the sense of passing to the commutant), in which case the cyclicity passes to the separating property of Ω with respect to $\mathcal{A}(\mathcal{O}')$. Referring to its use, the separating property is often called the *field-state relation*. The mathematical terminology is to say that the pair $(\mathcal{A}(\mathcal{O}), \Omega)$ is 'standard'. The large enough commutant required by the latter property is guaranteed by causality (the existence of a non-trivial causal disjoint $\mathcal{O}'$) and thus shows that causality is again responsible for this unexpected denseness property.

Of course the claim that somebody causally separated from us may provide us nevertheless with a dense set of states is somewhat perplexing especially if one compares it with the tensor factorization properties of good old Schrödinger QM with respect to an inside/outside separation via a quantization box.

If the naive interpretation of cyclicity/separability in the Reeh–Schlieder theorem leaves us with a feeling of science fiction (and for this reason as already mentioned has also justifiably attracted attention in philosophical quarters), the challenge for a theoretical physicist is to find an argument as to why, for all practical purposes, the situation nevertheless remains similar to QM. This amounts to the fruitful question of which vectors among the dense set of state vectors can be really produced with a controllable expenditure (of energy); a problem from which Haag and Swieca started their investigation. In QM this question was not that interesting, since the localization at a given time via support properties of wavefunctions leads to a tensor product factorization of inside/outside so that the inside state vectors are evidently never dense in the whole space and the 'particle behind the moon paradox' does not occur.

Later we will see that most of the very important physical and geometrical information is encoded into features of dense domains, in fact the aforementioned modular theory explains this deep relation between operator domains of the Tomita S and spacetime geometry. As mentioned before the individuality of the various S-operators is only the difference in domains, since all of them act as $SA\Omega = A^*\Omega$, $A \in \mathcal{A}(\mathcal{O})$

For the case at hand the reconciliation of the paradoxical aspect [34] of the Reeh–Schlieder theorem with common sense has led to the discovery of the physical relevance of *localization with respect to phase space in LQP*, i.e. the understanding of the *size of degrees of freedom* in the set: (notation $\boldsymbol{H} = \int E\, \mathrm{d}P_E$)

$$P_E\mathcal{A}(\mathcal{O})\Omega \quad \text{is compact} \tag{51}$$

$$P_E\mathcal{A}(\mathcal{O})\Omega \quad \text{or} \quad \mathrm{e}^{-\beta \boldsymbol{H}}\mathcal{A}(\mathcal{O})\Omega \quad \text{is nuclear.} \tag{52}$$

The first property was introduced by Haag and Swieca (as reviewed in [1]), whereas the second more refined statement (and similar nuclearity statements involving modular operators of local regions instead of the global Hamiltonian) which is saturated by QFT (i.e. cannot be improved) and is easier to use, is a later result of Buchholz and Wichmann [39]. It should be emphasized that the LQP degrees of freedom counting of Haag–Swieca, which gives an infinite but still compact (and even nuclear) set of phase-space localized states, is different from the QM finiteness of degrees of freedom per phase used in some contemporary entropy calculations.

The map $\mathcal{A}(\mathcal{O}) \to \mathrm{e}^{-\beta H}\mathcal{A}(\mathcal{O})\Omega$ is only nuclear if the mass spectrum of LQP is not too accumulative in finite mass intervals; in particular, infinite towers of equal-mass particles

† This weird aspect should not be held against QFT, but rather be taken as indicating that localization by a piece of hardware in a laboratory is also limited by an arbitrary large but finite energy, i.e. is a 'phase-space localization' (see the subsequent discussion). In QM one obtains genuine localized subspaces without energy limitations.

are excluded (which then would cause the strange appearance of a maximal 'Hagedorn' temperature). The nuclearity ensures that a QFT, which was given in terms of its vacuum representation, also exists in a thermal state. An associated nuclearity index turns out to be the counterpart of the quantum mechanical Gibbs partition function [1, 40] and behaves in an entirely analogous way.

The peculiarities of the above degrees-of-freedom counting are very much related to one of the oldest 'exotic' and at the same time characteristic aspects of QFT, namely vacuum polarization. As first observed by Heisenberg, the partial charge

$$Q_V = \int_V j_0(x)\,\mathrm{d}^3x = \infty \tag{53}$$

diverges as a result of uncontrolled vacuum particle/antiparticle fluctuations near the boundary. For the free-field current it is easy to see that a better definition involving test functions, which smoothens the behaviour near the boundary and takes into account the fact that the current is a four-dimensional distribution which has no restriction to equal times, leads to a finite expression.

$$Q_R = \int j_0(x) f(x_0) g\left(\frac{\boldsymbol{x}}{R}\right) \mathrm{d}^s x \tag{54}$$

where f and g are test functions of compact support with $\int f(x_0)\,\mathrm{d}x_0 = 1$ and $g(\vec{x}) = 1$ for $|\vec{x}| < 1$ and $g(\vec{x}) = 0$ $|\vec{x}| > 1 + \delta$. The vectors $Q_R\Omega$ only converge weakly for $R \to \infty$ on a dense domain. Their norms, however, diverge as [32]

$$\begin{aligned}(Q_R\Omega, Q_R\Omega) &\leqslant \text{constant} \times R^{s-1} \\ &\sim \text{area}.\end{aligned} \tag{55}$$

The surface-layer character of this vacuum polarization is reflected in this area behaviour together with the original divergence (53) for fixed R and $\delta \to 0$.

The algebraic counterpart is the so-called 'split property', namely the statement [1] that if one leaves between say the double-cone (the inside of a 'relativistic box') observable algebra $\mathcal{A}(\mathcal{O})$ and its causal disjoint (its relativistic outside) $\mathcal{A}(\mathcal{O}')$ a 'collar' (geometrical picture of the relative commutant) $\mathcal{O}_1' \cap \mathcal{O}$, i.e.

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}_1) \quad \mathcal{O} \ll \mathcal{O}_1 \quad \text{properly} \tag{56}$$

then it is possible to construct in a canonical way a type I tensor factor $\mathcal{N}$ which extends in a 'fuzzy' manner into the collar $\mathcal{A}(\mathcal{O})' \cap \mathcal{A}(\mathcal{O}_1)$, i.e. $\mathcal{A}(\mathcal{O}) \subset \mathcal{N} \subset \mathcal{A}(\mathcal{O}_1)$. With respect to $\mathcal{N}$ the Hilbert space factorizes, i.e. as in QM; there are states with no fluctuations (or no entanglement) for the 'smoothed' operators in $\mathcal{N}$. Whereas the original vacuum will be entangled from the box point of view, there also exists a disentangled product vacuum on $\mathcal{N}$. The algebraic analogue of a smoothing of the boundary by a test function is the construction of a factorization of the vacuum with respect to a suitably constructed type I factor algebra which uses the above collar extension of $\mathcal{A}(\mathcal{O})$. It turns out that there is a canonical, i.e. mathematically distinguished factorization, which lends itself to define a natural 'localizing map' Φ and which has given valuable insight into an intrinsic LQP version of Noether's theorem [1], i.e. one which does not rely on quantizing classical Noether currents. It is this 'split inclusion' which allows us to bring back the familiar structure of pure states, tensor product factorization, entanglement and all the other properties at the heart of standard quantum theory and the measurement process. However, despite all the efforts to return to structures known from QM, the original vacuum retains its thermal (entanglement) properties with respect to all localized algebras, even with respect to the 'fuzzy' localized $\mathcal{N}$.

Let us collect in the following some useful mathematical definitions and formulae for 'standard split inclusions' [41].

Definition 3. *An inclusion $\Lambda = (\mathcal{A}, \mathcal{B}, \Omega)$ of factors is called a standard split if the collar $\mathcal{A}' \cap \mathcal{B}$ as well as $\mathcal{A}, \mathcal{B}$ together with Ω are standard in the previous sense, and if in addition it is possible to place a type I_∞ factor $\mathcal{N}$ between $\mathcal{A}$ and $\mathcal{B}$.*

In this situation there exists a canonical isomorphism of $\mathcal{A} \vee \mathcal{B}'$ to the tensor product $\mathcal{A} \bar{\otimes} \mathcal{B}'$ which is implemented by a unitary $U(\Lambda) : H_\Lambda \to H_1 \bar{\otimes} H_2$ (the 'localizing map') with

$$U(\Lambda)(AB')U^*(\Lambda) = A \bar{\otimes} B' \qquad A \in \mathcal{A} \qquad B' \in \mathcal{B}'. \tag{57}$$

This map permits us to define a canonical intermediate type I factor $\mathcal{N}_\Lambda$ (which may differ from $\mathcal{N}$ in the definition)

$$\mathcal{N}_\Lambda := U^*(\Lambda)(B(H_1) \otimes \mathbb{1}U)(\Lambda) \subset \mathcal{B} \subset B(H_\Lambda). \tag{58}$$

It is possible to give an explicit formula for this canonical intermediate algebra in terms of the modular conjugation $J = U^*(\Lambda)J_{\mathcal{A}} \otimes J_{\mathcal{B}}U(\Lambda)$ of the collar algebra $(\mathcal{A}' \cap \mathcal{B}, \Omega)$ [41]

$$\mathcal{N}_\Lambda = \mathcal{A} \vee J\mathcal{A}J = \mathcal{B} \wedge J\mathcal{B}J. \tag{59}$$

The tensor product representation gives the following equivalent tensor product representation formulae for the various algebras:

$$\begin{aligned} &\mathcal{A} \sim \mathcal{A} \otimes \mathbb{1} \\ &\mathcal{B}' \sim \mathbb{1} \otimes \mathcal{B}' \\ &N_\Lambda \sim B(H_\Lambda) \otimes \mathbb{1}. \end{aligned} \tag{60}$$

As explained in [41], the uniqueness of $U(\Lambda)$ and N_Λ is achieved with the help of the 'natural cones' $\mathcal{P}_\Omega(\mathcal{A} \vee \mathcal{B}')$ and $\mathcal{P}_{\Omega\otimes\Omega}(\mathcal{A} \otimes \mathcal{B}')$. These are cones in Hilbert space whose position in H_Λ together with their facial subcone structures pre-empt the full algebra structure on a spatial level. The corresponding marvellous theorem of Connes [46] goes far beyond the previously mentioned state vector/field relation.

Returning to our physical problem, we have succeeded in finding the right analogue of the QM box. In contrast to the causally closed local type III algebras with their sharp lightcone boundaries ('quantum horizons'), the 'fuzzy box' type I factor N_Λ permits all the structures we know from QM: pure states, inside/outside tensor factorization, (dis)entanglement, etc with one exception: the vacuum is highly entangled in the tensor product description; the modular group of the state $\omega|_{\mathcal{A}\bar{\otimes}\mathcal{B}'}$represented in the tensor product natural cone $P_{\Omega\otimes\Omega}(\mathcal{A}\bar{\otimes}\mathcal{B}')$ is not the tensor product of the modular groups of $\mathcal{A}$ and $\mathcal{B}'$, whereas the modular conjugation J acts on the tensor product cone as $J_{\mathcal{A}}\bar{\otimes}J_{\mathcal{B}}$ (since the restriction $\omega|_{\mathcal{A}\bar{\otimes}\mathcal{B}'}$ is faithful). Note also that the restriction of the product state $\omega \otimes \omega$ to $\mathcal{B}$ or $\mathcal{B}'$ is not faithful, respectively, cyclic on the corresponding vectors and therefore the application of those algebras to the representative vectors $\eta_{\omega\otimes\omega}$ yields non-trivial projectors (e.g. $P_\Lambda = U^*(\Lambda)B(H_1)\bar{\otimes}1U(\Lambda)$).

Since the fuzzy box algebra N_Λ is of quantum mechanical type I, we are allowed to use the usual trace formalism based on the density matrix description, i.e. the vacuum state can be written as a density matrix ρ_Ω on $\mathcal{N}_\Lambda$ which leads to a well defined von Neumann entropy

$$(\Omega, A\Omega) = \operatorname{tr} \rho_\Lambda A \qquad A \in \mathcal{A} \tag{61}$$

$$S(\rho_\Lambda) = -\operatorname{tr} \rho_\Lambda \log \rho_\Lambda \tag{62}$$

but this is not sufficient to determine ρ_Λ, which is needed for the von Neumann entropy of the fuzzy box $S(\rho_\Lambda)$. If we were able to compute the unitary representative $\Delta^{it}_{\mathcal{N}_\Lambda}$ of the modular group of the pair $(\mathcal{N}_\Lambda, \Omega)$ then we would also know ρ_Λ since the modular operator of a type I factor is known to be related to an unnormalized density matrix $\check{\rho}_\Lambda$ with $\rho_\Lambda = \frac{1}{\operatorname{tr}\check{\rho}_\Lambda}\check{\rho}_\Lambda$ through the tensor product formula on $H_1\bar{\otimes}H_2$

$$\Delta = \check{\rho}_\Lambda\bar{\otimes}\check{\rho}_\Lambda^{-1}. \tag{63}$$

Actually, there are several intuitively equivalent definitions of localization entropy [47]. Among those the most convenient one seems to be the relative entropy of the vacuum ω with respect to the split vacuum $\omega \times \omega$. The relative entropy of a von Neumann algebra $\mathcal{M}$ of one faithful state ω_1 with respect to ω_2 uses the relative modular operator [1] $\Delta_{\omega_1,\omega_2}$,

$$S(\omega_1|\omega_2)_M = -\left\langle \log \Delta_{\omega_1,\omega_2}\right\rangle. \tag{64}$$

Kosaki [48] was able to convert this (in the most general setting) into a variational formula

$$\begin{aligned} &S(\omega_1|\omega_2)_{\mathcal{M}} = sup \int_0^1 \left[\frac{\omega(1)}{1+t} - \omega_1(y^*(t)y(t)) - \frac{1}{t}\omega_2(x^*(t)x(t))\frac{\mathrm{d}t}{t}\right] \\ &x(t) = 1 - y(t),\ x(t) \in \mathcal{M} \end{aligned} \tag{65}$$

where in our case $\omega_1 = \omega \times \omega$, $\omega_2 = \omega$, $\mathcal{M} = \mathcal{A} \vee \mathcal{B}'$.

Despite the very clear conceptual setting of this split entropy, it is difficult to obtain good estimates for this entropy, not to mention exact calculations. As for the above partial charges (55) one expects a surface behaviour, the quantum version of the Bekenstein–Hawking area law. An existing estimate shows that its increase for, for example, double cones is weaker than the spatial volume [47]. The most accessible situation for entropy calculations seems to be conformal QFT.

It seems that for double cones in conformal theories one can use the geometric aspects of the situation and perform an explicit calculation. This still needs to be carried out, but an outline of the strategy can be found in [7].

Ideas about localization entropy are quite inaccessible in perturbation theory because they require an intrinsic description in terms of a net of algebras, whereas for perturbation theory no description without the use of field coordinatizations is known. This has led to speculative remarks in the literature claiming the necessity of new degrees of freedom for the understanding of the area law.

5. Modular inclusions and intersections, holography

One of the oldest alternative proposals for canonical (equal-time) quantizations is the so-called light ray or lightfront (or $p \to \infty$ frame) quantization. The trouble with it is that it apparently inherits some of the short-distance problems from the canonical quantization. The latter is known to only makes sense for super-renormalizable interactions but not for strictly renormalizable ones, which lead to infinite multiplicative renormalizations. Let us ignore this for a moment and look at some additional problems of lightcone quantization which canonical equal-time quantization does not have. This is the apparent loss of the connection with local QFT; in fact, in none of the papers on lightcone quantization is it spelled out how to return to a local QFT. The problem of lightfront-restricted free fields was studied rigorously in [42], but in the interacting case the reconstruction of the local theory from that on the lightcone (which may be called the holographic reconstruction) is a serious problem indeed.

Our modular inclusion techniques in section 2 suggested that for massive (and massless for $d \neq 1+1$) theories the wedge algebra and the chiral lightfront algebra are identical

$$A(W) = A(\mathbb{R}_{>}). \tag{66}$$

Since this is a consistent property which is fulfilled by all known quantum field-theoretic models, we will be focus our interests on theories which obey this 'characteristic shadow' property and leave open the question of whether this property is a consequence of the standard physical requirements of AQFT. We already mentioned in the same section that the chiral algebra really should be thought of as the 'transversally unresolved lightfront algebra'. However, since the use of a lightfront notation like $A(\mathbb{R}_{>}^{d-1})$ could suggest the wrong idea that one deals with a full lightfront net, we prefer the light ray notation since it does not count any localization-wise unresolved dimensions. If we just refer to the global algebras and not to their local (sub)net structure, then all three objects are equal and there could be no confusion.

The rigorous construction of a chiral net for $\mathcal{A}(\mathbb{R}_{>})$ indicated in the section 3 will now be presented in more detail within its natural setting of modular inclusions [35].

One first defines an abstract modular inclusion in the setting of von Neumann algebras. There are several types of inclusions which have received mathematical attention†. An inclusion of two factors $\mathcal{N} \subset \mathcal{M}$ is called (+ half-sided) modular if the modular group $\Delta_{\mathcal{M}}^{it}$ for $t < 0$ transforms $\mathcal{N}$ into itself (compression of $\mathcal{N}$)

$$A\,\mathrm{d}\Delta_{\mathcal{M}}^{it}\mathcal{N} \subset \mathcal{N}. \tag{67}$$

We assume that $\cup_t A\,\mathrm{d}\Delta_{\mathcal{M}}^{it}\mathcal{N}$ is dense in $\mathcal{M}$ (or that $\cap_t \Delta_{\mathcal{M}}^{it}\mathcal{N} = \mathbb{C}\cdot 1$). This means, in particular, that the two modular groups $\Delta_{\mathcal{M}}^{it}$ and $\Delta_{\mathcal{N}}^{it}$ generate a two-parametric group of (translations, dilations) in which the translations have positive energy [13]. Let us now look at the relative commutant as done, for example, in the appendix of [43].

Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be modular with non-trivial relative commutant. Then look at the subspace generated by relative commutant $H_{red} \equiv \overline{(\mathcal{N}' \cap \mathcal{M})\Omega} \subset H$. The modular groups to $\mathcal{N}$ and $\mathcal{M}$ leave this subspace invariant: $\Delta_{\mathcal{M}}^{it}$, $t < 0$ maps $\mathcal{N}' \cap \mathcal{M}$ into itself by the inclusion being modular. Look at the orthogonal complement of H_{red} in H. This orthogonal complement is mapped into itself by $\Delta_{\mathcal{M}}^{it}$ for positive t. Let ψ be in that subspace, then

$$\left\langle \psi, \Delta_{\mathcal{M}}^{it}(\mathcal{N}' \cap \mathcal{M})\Omega \right\rangle = 0 \qquad \text{for} \quad t > 0. \tag{68}$$

Analyticity in t then gives the vanishing for all t.

Due to Takesaki's theorem [3], we can restrict $\mathcal{M}$ to H_{red} using a conditional expectation to this subspace defined in terms of the projector P onto H_{red}. Then

$$E(\mathcal{N}' \cap \mathcal{M}) \subset \mathcal{M}|_{\overline{(\mathcal{N}' \cap \mathcal{M})\Omega}} = E(\mathcal{M}) \tag{69}$$

$$E(\cdot) = P \cdot P \tag{70}$$

is a modular inclusion on the subspace H_{red}. $\mathcal{N}$ also restricts to that subspace and this restriction is obviously in the relative commutant of $E(\mathcal{N}' \cap \mathcal{M}) \subset E(\mathcal{M})$. Moreover, using arguments as above it is easy to see that the restriction is cyclic with respect to Ω on this subspace. Therefore, we arrive at a reduced modular 'standard inclusion'

$$(E(\mathcal{N}) \subset E(\mathcal{M}), \Omega). \tag{71}$$

Standard modular inclusions are known to be isomorphic to chiral conformal field theories [35].

† In addition to the split inclusion used in the previous section, there are the famous Jones inclusions, whose characteristic property is the existence of conditional expectations. Their domain in particle physics is in the area of charge fusion and internal symmetry.

This theorem and its extension to modular intersections leads to a wealth of physical applications in QFT, in particular, in connection with 'hidden symmetries' symmetries which are of purely modular origin and have no interpretation in terms of quantized Noether currents [23, 43]. The modular techniques unravel new structures which are not visible in terms of field coordinatizations. Holography and problems of degrees of freedom counting (phase space in LQP) as well as the issue of localization entropy are other examples.

Let us briefly return to applications for $d = 1 + 1$ massive theories. It is clear that in this case we should use the two modular inclusions which are obtained by sliding the (right-hand) wedge into itself along the upper/lower light ray horizon. Hence we chose $\mathcal{M} = \mathcal{A}(W)$ and $\mathcal{N} = \mathcal{A}(W_{a_+})$ or $\mathcal{N} = \mathcal{A}(W_{a_-})$ where $W_{a\pm}$ denote the two upper/lower light like translated wedges $W_{a\pm} \subset W$. As explained in section 2 following [30] and mentioned above, we do not expect the appearance of a non-trivial subspace (i.e. we expect $P = 1$) in the action of the relative commutants onto the vacuum

$$\begin{aligned} &\mathcal{A}(I(0, a_\pm)) \equiv A(W_{a_\pm})' \cap \mathcal{A}(W) \\ &\overline{\mathcal{A}(I(0, a_\pm)}\Omega = H \end{aligned} \tag{72}$$

where the notation indicates that the localization of $\mathcal{A}(I(0, a_\pm))$ is thought of as the piece of the upper/lower light ray interval between the origin and the endpoint $a_\pm$.

From the standardness of the inclusion one obtains according to the previous discussion an associated conformal net on the line, with the following formula for the chiral conformal algebra on the half-line:

$$\mathcal{A}_\pm(R_>) \equiv \bigcup_{t \geqslant 0} A\, \mathrm{d}\Delta_{\mathrm{W}}^{it}\, (\mathcal{A}(I(0, a_\pm))) \subseteq \mathcal{A}(W). \tag{73}$$

We expect the equality sign to hold

$$\mathcal{A}_\pm(R_>) = \mathcal{A}(W) \tag{74}$$

but our argument was tied to the existence of PFGs since as a result of their mass-shell structure

$$\begin{aligned} F(\hat{f}) &= \int Z(\theta) f(\theta)\, \mathrm{d}\theta \\ &= F_{\mathrm{res}}(\hat{f}_{\mathrm{res}}) \end{aligned} \tag{75}$$

where the notation 'res' indicates the corresponding generators in light ray theory which are identical in rapidity space and only differ in their x-space appearance. This is a significant strengthening of the cyclicity property $\overline{\mathcal{A}_\pm(R_>)\Omega} = \overline{\mathcal{A}(W)\Omega}$ for the characteristic data on one light ray. The argument is word for word the same in higher spacetime dimensions, since the appearance of transversal components (which have no influence on the localization) in addition to θ do not modify the argument. One would think that the inference of PFG generators can even be disposed of and the equality should follow from the standard causal shadow property of QFT in the form

$$\mathcal{A}(W) = A(R_>^{(\alpha)}) \tag{76}$$

where $R_>^{(\alpha)}$ is a spacelike positive half-line with inclination α with respect to the x-axis. The idea is that if this relation remained continuous for $R_>^{(\alpha)}$ approaching the light ray ($\alpha = 45°$) would then lead to the desired equality. We believe that the relation (76) for massive theories, which will be called the 'characteristic shadow property', is a general consequence of the standard causal shadow property (the identity of $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}'')$, where $\mathcal{O}''$ is the causal completion of the convex spacelike region $\mathcal{O}$) in any spacetime dimension.

Theories with the characteristic shadow property are the objects of light ray folklore. The present conceptually more concise approach explains why the light ray quantization in the presence of interactions is basically non-local which significantly restricts its unqualified physical use. The reason is that although the half-line algebra is equal to the wedge algebra (since all rays of forward lightcone propagation which pass through the upper/lower half light ray $R_>$ have passed or will pass through W), the locality on the light ray cannot be propagated into the wedge (the strips inside the wedge subtended from an interval I on the light ray by the action of the opposite light ray translation are for massive theories not outside the propagation region of the complement of I). Only for the half-line itself does one obtain a two-dimensional shadow region, namely the wedge region. If one uses both lightcones then it is possible to reconstruct a causal $d = 1+1$ net by intersections. This construction uses the two-dimensional translation group on the wedge and the ensuing double-cone relative commutants. Note that in order to achieve this with parity-reflected half-lines of light rays, one needs the relative position of the two half-line light ray algebras relative to each other in the common space H. In fact one shifted right light ray chiral algebra together with its parity-reflected image is equal to the union of two opposite spacelike separated wedge algebras. The reflected light ray algebra may also be replaced by the algebra on the left-hand extension of the original light ray since both create the same left wedge algebra. However, the natural net structure of that algebra is very non-local with respect to that of the parity-reflected one. This prevents its use in the construction of the two-dimensional net from shifts and geometric intersections on one light ray. An algebra localized in an interval on one light ray corresponds to a completely spread out algebra on the other ray. The modular group of each light ray interval is geometric. This agrees with the qualitative behaviour one expects for the modular group of the double cone in a massive theory [23] near the causal horizon. Note that the relative non-locality of the chiral conformal theories is also necessary in order to be consistent with a massive situation. The chiral conformal field theory contains the standard light ray translation with a gapless spectrum. However, this spectrum is not the physical one since in that chiral theory there exists yet another non-locally acting translation and it is the spectrum of the product of the two generators P_+P_- which gives the physical mass. Hence chiral conformal theories constitute a multipurpose tool in LQP. This is why they can serve as 'holographic' pieces for the construction of massive $d = 1 + 1$ theories. So with just one light ray and two translations, one acting locally and the other non-locally, one ray one can already reconstruct the full $d = 1 + 1$ net. Later we will see that this is enough to understand the localization entropy which turns out to have the surface behaviour first observed in the context of classical localization behind black hole horizons by Bekenstein.

Because of the transversal extension, the holography in terms of one-dimensional chiral conformal theories is more complicated for higher dimensions. There one needs a family of chiral conformal theories which is obtained from 'modular intersections'. Rather than associating the chiral conformal theory with a light ray, it is more appropriate to associate it with the transverse space of the wedge which contains the light ray, i.e. with the lightfront. A family of lightfront algebras is obtained by applying L-boosts to the standard wedge W, which tilts W around one of its defining light rays, so that the transversal degeneracy of the modular inclusion is partially destroyed (in $d = 1 + 2$ it would be completely destroyed). In this way one obtains a fan-like ordered family of wedges corresponding to a family of chiral conformal theories whose relative position within the original Hilbert space contains all the information necessary in order to reconstruct the original (massive) theory. A detailed and rigorous account of this construction will be given in a future paper. Here we will only mention some analogies to the above light ray situation. The process of tilting by applying a family of boost transformations which leave the common light ray invariant is described by unitary transformations of one chiral conformal theory into another. Each single one, according to the higher-dimensional characteristic shadow property, is equal to a wedge algebra. Knowing the position of a finite number of such chiral conformal theories with respect to each other (the number increases with increasing spacetime dimensions), determines the relative position of a finite number of wedge algebras ($\simeq$ chiral conformal QFTs) which according to the previous remarks is sufficient to reconstruct the original net (the blow-up property in [43]). As previously mentioned, in the $d = 1 + 1$ case the second light ray can be thought of as obtained from the first one by a unitary parity reflection (assuming that the theory is parity invariant). All the finitely many chiral conformal field theories are unitarily equivalent (either by parity or by L-boosts); the important physical information is contained in their relative position within the same Hilbert space. The terminology 'scanning by a finite family of chiral conformal theories' is perhaps more appropriate for this construction of higher-dimensional theories [43, 50].

It has been shown elsewhere [50] that the modular inclusion for two wedges gives rise to two reflected eight-parametric subgroups of the 10-parametric Poincaré group which contain a two-parametric transversal Galilean subgroup of the type found by formal lightfront quantization arguments [49]. All of these considerations show the primordial role of the chiral conformal QFT as a building block for the higher-dimensional QFTs.

There is another much more special kind of holography in which an isomorphism of a massive QFT in $d + 1$ dimensions to a conformal d-dimensional theory is the focus of interest. This isomorphism appears in Rehren's solution [19] of Maldacena's conjecture about the existence of a holographic relation of quantum matter in a $(d + 1)$-dimensional anti- de Sitter spacetime with that in a d-dimensional conformal QFT. This type of holography has not been observed outside the anti-de Sitter spacetime and since it is an isomorphism to a conformal theory, the degrees of freedom are not really reduced in the sense of 't Hooft [44], as was the case in the previous holography via light ray reduction. The Maldacena–Witten (M–W) holography is apparently of importance within the development of string theory, in fact, its protagonists believe that it contains information about a possible message about the quantum gravity of string theory. Within the present AQFT setting its main interest is that it requires the field-coordinatization-free point of view in its strongest form; whereas in most problems of QFT there exist appropriate field coordinatizations which often facilitate calculations, the M–W isomorphism defined in rigorous terms by Rehren is not pointlike and has no description in terms of fields outside its algebraic version. In contradistinction to the light ray holography which happens at the causality horizon (lightfront boundary) of modular localization (or its classical Killing counterpart in the case of black holes) the AdS holography takes place at the boundary at infinity.

A very simple presentation in the spirit of Rehren's approach which takes into account the covering of the relevant spaces can be found in [51].

6. Comparison with string theory

As mentioned in the introduction, historically string theory originated from the attempt to understand and implement the issue of crossing symmetry of the S-matrix. Without the intervention of QFT it was difficult to combine unitarity and crossing symmetry into a manageable formalism. It came as somewhat of a surprise that by assuming an additional stronger form of crossing called 'duality' one actually could obtain the dual-model formalism. Duality was an idea of entirely phenomenological origin which consisted in the hypothesis that crossing can already hold if one only restricts ones attention to (Reggeized) one-particle states ('particle democracy'). There was no theoretical support from QFT, nevertheless the very appealing form of duality by Veneziano led eventually to string theory. However, whereas the content of QFT can be separated from the perturbative formalism and cast into a totally intrinsic form which is strongly related to its underlying principles, string theory leaves a lot to be desired on conceptual aspects and remained a collection of prescriptions. In particular, string theorists have not been able to successfully address the issue of locality of operators and localization of states which are absolutely crucial properties on which any particle physics theory stands and falls and which are even indispensable for the physical interpretation of its formalism [34]. The formal basis of string theory is a kind of momentum space 'engineering' rather than a conceptual spacetime analysis. The latter remained within the realm of quasiclassical physics using geometrical pictures with some fluctuation caused fuzziness, i.e. pictures which in the setting of quantum theory fall behind Heisenberg's dictum that positions and momenta are not properties of the electron but are characteristics of the events involving interactions with a measurement apparatus which causes the factualization of potentialities. Related to this is the fact that the word scattering theory has an entirely different meaning in both areas. Whereas in QFT it is an asymptotic relation to free fields for whose derivation spacelike locality is absolutely essential, in string theory its use in the sense of the $0/\infty$ behaviour of the analytically continued source space conformal field theory in the complex plane has nothing to do with any standard scattering concept of physical particles. Whereas all important ideas in QFT have been tested outside quasiclassical or perturbative settings at least in $d = 1 + 1$ interacting theories, this is not the case in string theory. For example, the Klein–Kaluza mechanism for the conversion of spacetime into inner symmetry which is a (semi)classical idea has never been tested in a full QFT. Since the physical origin of internal symmetries is closely related to particle/field statistics†, there is some subtle problem with the Klein–Kaluza mechanism in QFT away from the quasiclassical pictures of functional integrals.

Another problematic point is the intrinsic meaning of 'stringiness' in the form of an infinite tower of particles with an oscillator-like mass spectrum. As long as mass spectra do not accumulate (by increase of multiplicities) too densely, they are compatible with the phase-space structure of QFT and lead to reasonable thermal behaviour, i.e. the pathological situation of a finite Hagedorn temperature can presumably also be avoided in string theory. However, it is not known to me how one can distinguish an infinite collection of resonances, i.e. poles in the second Riemann sheet (since presumably in string theory most of the particles in the tower are unstable through higher-order (higher-generi) interactions as would be the case in Feynman theory). I do not know of any theorem in QFT which forbids such a resonance situation and therefore I do not understand the meaning of stringiness. Extended objects can also exist in QFT built on perfect local observables; in fact, the superselection theory even demands in some cases the existence of non-compactly localized objects which intertwine between inequivalent representations of perfectly local observable algebras, examples are the carriers of braid group statistics in $d = 1+2$ dimensions are necessarily extended along semi-infinite spacelike strings. So it is very questionable whether there exists an intrinsic meaning of stringiness.

The relation of string theory with the wedge-localization approach to QFT presented in this paper goes only via the common historical root of the S-matrix theory of the 1960s and basically consists in the claim that both theories are ultraviolet finite. In fact, the on-shell nature of wedge algebras as exemplified by the modular wedge-localization equation (35) provides a field-theoretic link for the S-matrix bootstrap and transports the ultraviolet finiteness of the

† The analysis of statistics from first principles leads rather directly to parastatistics in the sense of [1]. It is one of the great achievements of particle physics in the 1980s to show that this may always be converted into fermion/boson statistics + internal group symmetry where the latter can be computed from the structure of the structure of the causal observables.

latter into QFT. Although this finiteness is shared with string theory, the cause of it is very different. Whereas in string theory this finiteness results from the extension† of a string as an indecomposable state of matter, the modular approach to QFT is ultraviolet-finite in a much more radical and at the same time much more conservative way. The radical aspect is that by not using the inevitably lightcone singular-field coordinatizations in the actual construction but rather a net of algebras, the objects to which the bad short-distance behaviour and the ultraviolet divergences are attached have disappeared from the scene. They may be constructed at the end as local generators of the already constructed spacetime-indexed nets of algebras, but there they can no longer do any harm. The conservative aspect is that by taking this approach which requires a radically changed formalism, one remains in total harmony with the causality, spectral, and degrees of freedom principles which underlie QFT. The short-distance behaviour of the field approach is substituted by the non-triviality of intersections of algebras. This approach has already been tested in the bootstrap-formfactor constructions of $d = 1 + 1$ factorizable models. In $d = 1 + 3$ one expects that its perturbative version reproduces the renormalizable field theories and, in addition, reveals whether the frontiers of the standard approach (renormalizable/non-renormalizable) which appear in a purely formal way (power counting in auxiliary objects) are really the intrinsic formalism-independent frontiers defined by the physical principles of QFT. Massive gauge theories analysed from the slightly physical point of view [38] of self-interacting massive vector mesons nourish the suspicion that the intrinsic frontiers may be wider than those set by the standard perturbative power counting for interaction polynomials.

Both the modular wedge localization approach as well as string theory attribute a basic significance to chiral conformal theory, and both know the notion of holography. However, the use and the physical interpretation of these concepts is quite different. Whereas in AQFT chiral conformal theories are the building blocks of holographic images of higher-dimensional theories and therefore are positioned in the same Minkowski space, string theory places the chiral conformal data into an auxiliary source space and identifies the physical space as the target space of the fields in which they take their values. Related to this is, in fact, the notorious difficulty of defining a string field theory, a problem which is presumably related to the difficulty in separating the intrinsic conceptual content of string theory from its procedural prescriptions.

On the other hand, the modular approach has all the hallmarks of a conceptually based intrinsic formulation of local particle physics which makes it a candidate for an extension into the realm of interactions of the Wigner's representation theory of free particles which was the first totally intrinsic (independent of quantization) approach to relativistic quantum theory.

Note added in proof. Meanwhile there has been progress on the issue of particle versus field structure in conformal theories and on a time-like braid group structure behind the spectrum of anomalous scale dimensions [52].

Acknowledgments

A correspondence with K-H Rehren, and discussions with J Roberts and R Longo during a short visit of Rome university led to various improvements of the paper. I am particularly indebted to J Roberts for the invitation which made these discussions possible. Finally, I would like to thank my colleagues at the FU Andreas Fring, Michael Karowski, Hradch Babujian and Robert Schrader for some good questions and constructive suggestions.

† This means that the string of string theory is not an extended object in an otherwise local theory such as, for example, a Mandelstam string in gauge theory.

Appendix A. Some facts about modular theory

Definition 4. *A von Neumann algebra $\mathcal{A}$ (weakly closed operator sub-algebra of the full algebra B(H) on a Hilbert space H) is in 'standard position' with respect to a vector $\Omega \in H$, denoted as $(\mathcal{A}, \Omega)$, if Ω is a cyclic ($\overline{\mathcal{A}\Omega} = H$) and separating ($A\Omega = 0$, $A \in \mathcal{A}$ iff $A = 0$) vector for $\mathcal{A}$. In this situation Tomita defines the following involutive antilinear but unbounded operator (the Tomita involution S):*

$$SA\Omega := A^*\Omega \tag{A1}$$

where the star operation is the Hermitian conjugate in operator algebras. Its closeability property (as physicists we will use the same notation for the closure) is the prerequisite for the polar decomposition

$$S = J\Delta^{1/2} \tag{A2}$$

where the angular part J (the modular involution) is antiunitary with $J^2 = 1$ and Δ is unbounded positive and therefore leads to a unitary group Δ^{it}.

Theorem 5 (Tomita (1965), with significant improvements from Takesaki). *The modular involution maps $\mathcal{A}$ onto its von Neumann commutant $\mathcal{A}'$ in H:*

$$A\,\mathrm{d}J \cdot \mathcal{A} = \mathcal{A}'. \tag{A3}$$

The unitary Δ^{it} defines a 'modular' automorphism group by

$$A\,\mathrm{d}\Delta^{it} \cdot \mathcal{A} = \mathcal{A} \tag{A4}$$

(in analogy to a dynamical law for the algebra).

More details and references to the proof can be found in [1]. Actually, physicists have independently discovered some important properties of modular theory which later were incorporated by mathematicians into the Tomita–Takesaki theory. In fact, Haag *et al* [1] observed that the KMS property which Kubo, Martin and Schwinger just used a computational trick in order to avoid the calculation of traces in quantum statistical mechanics took on a fundamental conceptual role if one works directly in the thermodynamic limit of infinitely extended systems. A closely related independent discovery in their pursuit of physical-conceptual problems in quantum statistical mechanics which arise if one works directly in the thermodynamic limit [1]. As is well known, the Gibbs representation formula

$$\begin{aligned} &\langle A_V \rangle_\beta^{(V)} = \frac{\mathrm{tr}\, \mathrm{e}^{-\beta H_V} A_V}{\mathrm{tr}\, \mathrm{e}^{-\beta H_V}} \\ &A_V \in \text{ algebra of box-quantization} \end{aligned} \tag{A5}$$

ceases to make sense† for infinite volume open systems and the algebra changes its Murray–von Neumann type. Whereas in the quantization box it was type I, the open system algebra becomes type III_1 and the Gibbs formula passes to the KMS condition which is a cyclic relation for thermal correlation functions [1]. In the 1970s Haag and collaborators were able to derive the KMS condition directly from stability properties under local deformations and Pusz and Woronowicz found a direct link to the second law of thermodynamics [1]. These profound results were recently used for the derivation of thermal properties of quantum matter in an anti-de Sitter spacetime [4].

† In a box the bounded-below Hamiltonian acquires a discrete spectrum and $\mathrm{e}^{-\beta H}$ is of trace class ($\Omega_\beta = \mathrm{e}^{-\frac{1}{2}\beta H}$ is Hilbert–Schmidt.), a property which is lost in the infinite-volume limit.

The relation of modular theory with the Einstein causality of observables and locality of fields in QFT was made around 1975 in a series of papers by Bisognano and Wichmann (referred to in [1]). Specializing to wedge algebras $\mathcal{A}(W)$ generated by Wightman fields, they proved the following theorem.

Theorem 6. *The Tomita modular theory for the wedge algebra and the vacuum state vector* $(A(W), \Omega)$ *yields the following physical identifications:*

$$\Delta^{it} = U(\Lambda_W(2\pi t)) \qquad J = TCP \cdot U(R_x(\pi)). \tag{A6}$$

Here $\Lambda_W(\chi)$ denotes the boost (χ is the x-space rapidity) which leaves the wedge W invariant. If we choose the standard t–x wedge, then the rotation which aligns the TCP with Tomita's J is a rotation around the x-axis by an angle π.

Now I come to my own contributions which are of a more recent vintage [21]. They result from the desire to invert the Bisognano–Wichmann theorem, i.e. to use Tomita's modular theory for the actual construction (and classification) of (a net of) wedge algebras belonging to interacting theories with the final goal to intersect wedge algebras in order to obtain a net of compactly localized double-cone algebras. For the arguments which show that the particle physics properties, in particular the scattering matrix and formfactors of distinguished fields (conserved currents) can be abstracted from the net observables, I refer to [1, 36, 40]. If desired, the nets can also be coordinatized by more traditional pointlike fields and a rigorous derivation for chiral nets can be found in [33]. For the derivation of LSZ scattering theory one makes the assumption of the existence of a mass gap. With this one immediately realizes that, whereas the connected part of the Poincaré group is the same as that of the free incoming theory, the disconnected part containing time reversals, in particular the modular involution J for the wedge carry the full interaction

$$\Delta_W^{it} = \Delta_{W,\mathrm{in}}^{it} =: \mathrm{e}^{-\mathrm{i}Kt} \qquad J_W = S_{sc} J_{W,\mathrm{in}}. \tag{A7}$$

Here $J_{W,\mathrm{in}}$ refers to the Tomita involution (or TCP reflection) of the wedge algebra generated by the incoming free field. If the theory is not asymptotically complete (i.e. the vacuum is not cyclic with respect to the incoming fields) these relations have to be modified, but here we discard such pathologies for which no physical illustration exists. Since we do not want to tamper with historical notation, we have added a subscript to the S-matrix S_{sc} in order to distinguish it where necessary from Tomita's S. The modular 'Hamiltonian' K defined in the first equation (the boost generator = Hamiltonian of a particular uniformly accelerated Unruh observer) always has a symmetric instead of a one-sided spectrum.

The last relation (A7) is nothing but the TCP transformation law of the S-matrix rewritten in terms of modular objects associated with the wedge algebra. The above role of the S-matrix as a kind of relative modular invariant of the wedge algebra (relative to the free one) is totally characteristic for *local* quantum physics and has no counterpart in quantum mechanics.

Appendix B. Absence of PFGs for sub-wedge regions in theories with interactions

Theorem 7. *In interacting theories there exist no PFGs localized in subwedge regions. The wedge region is the smallest spacetime region for which PFGs in the presence of interactions are possible.*

For the proof† let us first assume that the spacetime localization region $\mathcal{O}$ of the would be PFGs is compact, e.g. a double cone. Let ϕ be an operator which is affiliated with $\mathcal{A}(\mathcal{O})$ which

† The proof is similar to that of the Jost–Schroer theorem in [5] and to that in [18].

means that on the domain of definition it commutes with all operators from the commutant $\mathcal{A}'(\mathcal{O})$. The PFG property of ϕ means ($\phi^{\#}$ denotes either ϕ or ϕ^*)

$$\begin{aligned} \phi^{\#}(x)\Omega &= \text{one-particle vector} \\ \phi^{\#}(x) &= U(x)\phi^{\#}U^*(x) \end{aligned} \tag{B1}$$

without any admixture of additional polarization contribution from higher particle configurations. As a result the vector satisfies the free-field equation in x. On the other hand, we have that $\left[\phi^{\#}(x), \phi^{\#}(y)\right] = 0$ for sufficiently large spacelike separations. Let us now look at the matrix elements

$$\left\langle \psi_2 \left| \phi^{\#}(x) \right| \psi_1 \right\rangle \tag{B2}$$

with say $\psi_2 \in \text{domain}(\phi^{\#})$ and choose ψ_1 from the dense set of state vectors which are localized in some region spacelike relative to $\text{loc}(\phi^{\#}(x))$. This is done by applying spacelike separated operators onto the vacuum $|\psi_1\rangle = A\,|0\rangle$. Since $\phi^{\#}(x)$ commutes with such operators we obtain

$$\left\langle \psi_2 \left| \phi^{\#}(x) \right| \psi_1 \right\rangle = \left\langle \psi_2 \left| \phi^{\#}(x) A \right| \Omega \right\rangle = \left\langle \psi_2 \left| A\phi^{\#}(x) \right| \Omega \right\rangle \tag{B3}$$

i.e. $\phi^{\#}(x)$ fulfils the free-field equation on a dense set of states in its domain. Since all affiliated operators are closeable, the operator itself fulfils the free-field equation. If we succeed to prove in addition that the commutator with itself is a c-number

$$\left[\phi^*(x), \phi(y)\right] = c(x-y)\,\mathbb{1} \tag{B4}$$

then we would have achieved our goal since it would follow that $\phi^{\#}(x)$ is a linear expression in terms of the particle creation and annihilation operator which contradicts the presence of an interaction. However, this last step follows almost literally the argument in the derivation of the Jost–Schroer theorem [5], the fact that the present ϕ has no well defined L-covariance does not matter. In the first step one shows that

$$\left[\phi^*(x), \phi(y)\right]|\Omega\rangle = c(x-y)\,|\Omega\rangle \tag{B5}$$

which requires the creation × creation contribution to vanish, i.e. $\left[\phi^{*(+)}(x), \phi^{(+)}(y)\right]|\Omega\rangle = 0$. For this one uses causality and the separate analyticity in x and y, which follows from the forward mass-shell support property. The generalization from a relation on the vacuum to a relation on a dense set of states is as before.

References

[1] Haag R 1992 *Local Quantum Physics* (Berlin: Springer)

[2] Borchers H-J 2000 On the revolutionization of quantum field theory by Tomitas modular theory *J. Math. Phys.* at press

[3] Takesaki M 1970 Tomita's theory of modular Hilbert algebras and its applications (*Lecture Notes in Mathematics* vol 128) (Berlin: Springer)
Stratila S 1981 *Modular Theory in Operator Algebras* (Turnbridge Wells: Abacus)

[4] Buchholz D, Florig M and Summers S J 1999 Hawking–Unruh temperature and Einstein causality in anti-de Sitter space-time *Class. Quantum Grav.* **17** 31
(Buchholz D, Florig M and Summers S J 1999 *Preprint* hep-th/9905178)

[5] Streater R F and Wightman A S 1964 *PCT, Spin and Statistics and All That* (Menlo Park, CA: Benjamin-Cummings)

[6] Bogoliubov N N, Logunov A A, Oksak A A and Todorov I T 1990 *Introduction to Axiomatic Quantum Field Theory* (Dordrecht: Kluwer)

[7] Schroer B 2000 Particle physics and QFT at the turn of the century: old principles with new concepts (an essay on local quantum physics) *J. Math. Phys.* to appear

[8] Heisenberg W 1946 *Z. Naturf.* **1** 608

[9] Chew G 1961 *The S-Matrix Theory of Strong Interactions* (New York: Benjamin-Cummings)
[10] Berg B 1977 *Nuovo Cimento* A **41** 58
[11] White A R 2000 The past and future of *S*-matrix theory *Preprint* hep-ph/0002303
[12] Bisognano J J and Wichmann E H 1975 *J. Math. Phys.* **16** 985
Bisognano J J and Wichmann E H 1979 *J. Math. Phys.* **17** 303
[13] Wiesbrock H-W 1993 *Commun. Math. Phys.* **157** 83
Wiesbrock H-W 1997 *Commun. Math. Phys.* **184** 683 (erratum)
Wiesbrock H-W 1997 *Lett. Math. Phys.* **39** 203
[14] Newton T D and Wigner E P 1949 *Rev. Mod. Phys.* **21** 400
[15] Schroer B, Truong T T and Weisz P 1976 *Phys. Lett.* B **63** 422
[16] Karowski M, Thun H J, Truong T T and Weisz P 1977 *Phys. Lett.* B **67** 321
[17] Zamolodchikov A B 1977 *Sov. Phys.–JETP Lett.* **25** 468
[18] Mund J 1998 *Math. Phys.* **43** 319
[19] Rehren K-H 1999 Algebraic holography *Preprint* hep-th/9905179
[20] Weinberg S 1995 *The Quantum Theory of Fields* vol I (Cambridge: Cambridge University Press)
[21] Schroer B 1999 *Ann. Phys., NY* **275** 190
[22] Brunetti R, Guido D and Longo R 2000 First quantization via BW property, in preparation
[23] Schroer B and Wiesbrock H-W 2000 *Rev. Math. Phys.* **12** 139
[24] Bratteli O and Robinson D W 1979 *Operator Algebras and Quantum Statistical Mechanics* vols I, II (Berlin: Springer)
[25] Karowski M and Weisz P 1978 *Nucl. Phys.* B **139** 455
[26] Zamolodchikov A B 1989 *Int. J. Mod. Phys.* A **1** 1235
[27] Fring A 1996 *Int. J. Mod. Phys.* A **11** 1337
[28] Smirnov F A 1989 *Int. J. Mod. Phys.* A **4** 156
Our operator notation is closer to that of
Lashkevich M Yu 1994 Sectors of mutually local fields in integrable models of quantum field theory *Preprint* hep-th/9406118
[29] Babujian H, Fring A, Karowski M and Zapletal A 1999 *Nucl. Phys.* B **538** 53
[30] Guido D, Longo R, Roberts J E and Verch R 1999 Charged sectors, spin and statistics in quantum field theory on curved spacetimes *Preprint* math-ph/9906019 and references therein
[31] Borchers H-J, Buchholz D and Schroer B 2000 Polarization-free generators and the *S*-matrix *Preprint* hep-th/0003243
[32] Buchholz D, Doplicher S, Longo R and Roberts J H 1992 *Rev. Math. Phys.* **SI1** 49
[33] Fredenhagen K and Joerss 1996 *Commun. Math. Phys.* **176** 541
[34] Schroer B 1999 *J. Phys. A: Math. Gen.* **32** 5937
[35] Guido D, Longo R and Wiesbrock H-W 1998 *Commun. Math. Phys.* **192** 217
[36] Buchholz D and Haag R 2000 The quest for understanding in relativistic quantum physics *J. Math. Phys.* at press
[37] Rob Clifton and Hans Halvorson 2000 Entanglement and open systems in algebraic quantum field theory, University of Pittsburgh *Preprint* quant-ph/0001107
[38] Duetsch and Schroer B 2000 *J. Phys. A: Math. Gen.* **33** 4317
(Duetsch and Schroer B 1999 Massive vectormesons and gauge theory *Preprint* hep-th/9906089)
[39] Buchholz D and Wichmann E H 1986 *Commun. Math. Phys.* **106** 321
[40] Buchholz D 1998 Current trends in axiomatic quantum field theory *Preprint* hep-th/9811233
[41] Doplicher S and Longo R 1984 *Invent. Math.* **75** 493
[42] Dimock J 1998 Locality in free string theory *Preprint* mp-arc 98-311
[43] Schroer B and Wiesbrock H-W 2000 *Rev. Math. Phys.* **12**
[44] ’t Hooft G 1993 *Salam-Festschrift* ed A Ali *et al* (Singapore: World Scientific) p 284
[45] Doplicher S and Roberts J E 1990 *Commun. Math. Phys.* **131** 51
[46] Connes A 1974 *Ann. Inst. Fourier* **126** 121
[47] Narnhofer H 1994 *The State of Matter* ed M Aizenman and H Araki (Singapore: World Scientific)
[48] Kosaki H 1986 *J. Operator Theory* **16** 335
[49] Susskind L 1995 *J. Math. Phys.* **36** 6377
[50] Schroer B and Wiesbrock H-W 2000 *Rev. Math. Phys.* **12**
[51] Schroer B 1999 Facts and fictions about anti de Sitter spacetimes with quantum matter *Preprint* hep-th/9911100
[52] SchroerB 2000 *Preprints* hep-th/0005010 and hep-th/0005134